DMP Class Test

Solutions
23 October 2024

1. We prove the equivalent statement that k^2+k is even for each integer k.

   In fact k^2+k can be written as $k(k+1)$, and either k or $k+1$ must be even. When multiplying an even number with an integer, the result is still even.

2. The proof is by strong induction. Let $P(n)$ be the statement "$e_n = 3^{f_n}$".

   Base cases: $e_0 = 1 = 3^0 = 3^{f_0}$. So $P(0)$ holds. Moreover, $e_1 = 3 = 3^1 = 3^{f_1}$. So $P(1)$ holds too.

   Induction step: Assuming, for some $k \geq 1$, that $P(j)$ holds for all j with $0 \leq j \leq k$, we must obtain $P(k+1)$. So we assume that $e_j = 3^{f_j}$ for all j with $0 \leq j \leq k$. Below we use this assumption for the two values $j = k$ and $j = k-1$.

   Now $e_{k+1} = e_k \cdot e_{k-1} = 3^{f_k} \cdot 3^{f_{k-1}} = 3^{f_k + f_{k-1}} = 3^{f_{k+1}}$. This is $P(k+1)$.

   Thus, by strong induction, $P(n)$ holds for all $n \geq 0$.

3. To prove that $(A-B)\cup(B-C)\cup(C-A) = (B-A)\cup(C-B)\cup(A-C)$ by the element method, first suppose that $x \in (A-B)\cup(B-C)\cup(C-A)$. There are three cases to consider. For reasons of symmetry, I need to consider only the case that $x \in A-B$. The other two cases, that $x \in B-C$ or $x \in C-A$, necessary proceed in the same way — one can see that this must be so by cyclicly rotating the roles of A, B and C.

   Assuming $x \in A-B$, we obtain that $x \in A$ and $x \notin B$. Now we make a further case distinction, depending on whether $x \in C$.

   In case $x \in C$, given that $x \notin B$, we have $x \in C-B$, and thus $x \in (B-A)\cup(C-B)\cup(A-C)$.

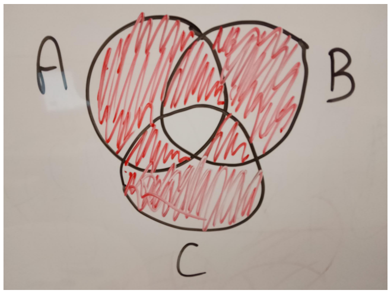
   In case $x \notin C$, given that $x \in A$, we obtain $x \in A-C$, and thus $x \in (B-A)\cup(C-B)\cup(A-C)$.

   So in all cases $x \in (B-A)\cup(C-B)\cup(A-C)$.
   It follows that $(A-B)\cup(B-C)\cup(C-A) \subseteq (B-A)\cup(C-B)\cup(A-C)$

   The other direction, that $(B-A)\cup(C-B)\cup(A-C) \subseteq (A-B)\cup(B-C)\cup(C-A)$ follows by symmetry. In fact, this statement is seen to be equivalent to the one we proved above by exchanging the roles of A and B.

   Together, we obtain $(A-B)\cup(B-C)\cup(C-A) = (B-A)\cup(C-B)\cup(A-C)$.

   And here is the Venn diagram:

4. First simplify the equation by dividing all coefficients by 7:

$$9 \cdot x - 7 \cdot y = 8.$$

Let's work modulo 7.
$9 \cdot x - 0 \equiv 8 \equiv 15 \equiv 22 \equiv 29 \equiv 36 \pmod 7$. So $x = 4 \pmod 7$.
Abandoning the modulo arithmetic, we find $x = 4 + k \cdot 7$. Hence $9 \cdot (4 + k \cdot 7) - 7 \cdot y = 8$.
$7 \cdot y = 9 \cdot 7 \cdot k + 36 - 8 = 9 \cdot 7 \cdot k + 28$, so $y = 9 \cdot k + 4$.
The pair $(x, y) = (7k + 4, 9k + 4)$ turns out to be a valid solution for each integer k:
$9 \cdot (7k + 4) - 7 \cdot (9k + 4) = 9 \cdot 4 - 7 \cdot 4 = 2 \cdot 4 = 8$.
Moreover, this method is assured to produce all such valid solutions.

5. f is clearly injective, but not surjective, as there is no x with $f(x) = 0$ (or with $f(x) = 25$).
Thus f is not bijective either.
g is surjective, for each integer y can be obtained as $g(3(y - 7))$.
However, g is not injective, for $g(4) = g(5)$.
Thus g is not bijective either.
Since g is not injective, $f \circ g$ cannot be injective, regardless what f is: $f \circ g(4) = f \circ g(5)$.
Since f is not surjective, $f \circ g$ cannot be surjective, regardless what g is:
there is no x with $(f \circ g)(x) = 0$ (or with $(f \circ g)(x) = 25$).
Thus $f \circ g$ is not bijective either.
$(g \circ f)(x) = x + 8$. This function is (injective, surjective and) bijective.

6. For each real number $r \in \mathbb{R}$, define the function f_r by

$$f_r(x) = \begin{cases} 1 & \text{if } x = r \\ 0 & \text{otherwise} \end{cases}$$

For two different $r, s \in \mathbb{R}$ surely f_r and f_s are different functions. Thus we have found an injective mapping from the reals to the set S of functions from the reals to the integers. This implies that S is at least as large as $\mathbb{R}$. Since $\mathbb{R}$ is uncountable, so is S.
(Alternative argument: Let $T := \{f_r \mid r \in \mathbb{R}\}$. The above creates a bijection between $\mathbb{R}$ and T, so T is as large as $\mathbb{R}$, which is uncountable. Since $T \subseteq S$, the set S must be uncountable too.)