This homework runs from Thursday 19 September 2024 until 12 noon on Thursday 26 September 2024. Submission is to Gradescope Homework 1.

Questions marked with an asterisk $*$ may be a little harder than others. All are still within the course curriculum, though, and can be done using the methods taught in the study guides and textbook.

You should aim to write out solutions that someone who does not already know the answer could follow and understand.

Question 1

(a) Write down the negation of the following statement.

For all integers n, if n is odd then $(n^2 + 4)$ is prime

[*2 marks*]

(b) Use a counterexample to prove the statement in part (a) is false. [*2 marks*]

Question 2

Prove by contraposition that for any irrational number r its cube root $\sqrt[3]{r}$ is also irrational. [*4 marks*]

$*$ Question 3

Write out a proof that if a and b are integers then not all of a, $(a+b)$, and ab are odd. [*2 marks*]

Solution 1

(a) The negation is "There exists an integer n such that n is odd and $(n^2 + 4)$ is not prime". It would also be correct to say "...is composite" instead of "...not prime".

(b) For proof by counterexample we need to show the statement in part (a) is true: to find an odd integer n such that $(n^2 + 4)$ is composite. The following table tests this for small odd integers.

n	$(n^2 + 4)$	
1	5	is prime
3	13	is prime
5	29	is prime
7	53	is prime
9	85	is composite, $85 = 5 * 17$

So $n = 9$ is a counterexample and shows the original statement is false.

The table here isn't essential to the proof — we might simply guess at $n = 21$ and then observe that $21^2 + 4 = 445$ is a multiple of 5 and so that works as a counterexample. But the table is a methodical way to find a counterexample, and it's still essential to present the calculation that confirms it has the necessary property.

Solution 2

The contrapositive statement is that for any real number r, if $\sqrt[3]{r}$ is rational then r is rational. Suppose then then that $\sqrt[3]{r} = \frac{p}{q}$ for integers p and q with $q \neq 0$. Then $r = (\sqrt[3]{r})^3 = \frac{p^3}{q^3}$. Both p^3 and q^3 are integers, with $q^3 \neq 0$, and so r is rational as required.

* **Solution 3**

This can be proved both by contradiction and by cases.

To prove by contradiction, suppose that the statement is false: that there are in fact some integers a and b with all of a, $(a + b)$, and ab being odd. Then $b = ((a + b) - a)$ is even, and hence so is ab. But ab cannot be both odd and even, this is a contradiction, and the original statement must have been true.

In a similar way, if ab is odd then both a and b are odd which means $(a + b)$ is even: hence not all can be odd at the same time.

To prove by cases, consider the parity of a.

- If a is even then not all of a, $(a + b)$, and ab are odd: at least a and ab are even.
- If a is odd then we consider the parity of b as well.
  - If a is odd and b is even then ab is even and so not all of a, $(a + b)$, and ab are odd.
  - If a is odd and b is odd then $a + b$ is even and so not all of a, $(a + b)$, and ab are odd.

In all cases, not all of a, $(a + b)$, and ab are odd, and so the original statement is proved.

We can also draw up a table showing these results.

	If b is even:	If b is odd:
If a is even:	All of a, $(a+b)$ and ab are even	Both a and ab are even
If a is odd:	Only ab is even	Only $a+b$ is even

From this it is clear that a, $(a+b)$ and ab are never all odd.