This homework runs from Thursday 26 September 2024 until 12 noon on Thursday 3 October 2024. Submission is to Gradescope Homework 2.

Questions marked with an asterisk $*$ may be a little harder than others. All are still within the course curriculum, though, and can be done using the methods taught in the study guides and textbook.

You should aim to write out solutions that someone who does not already know the answer could follow and understand.

Please remember the good scholarly practice requirements of the University regarding work for credit. You can find guidance at the School page https://web.inf.ed.ac.uk/infweb/admin/policies/academic-misconduct. This also has links to the relevant University pages.

Question 1

(a) For which positive integers n is $3^n < n!$?

(b) Prove by induction your statement in (a).

[*3 marks*]

Question 2

Suppose a sequence of integers $a_0, a_1, a_2, \ldots$ is defined recursively as follows:

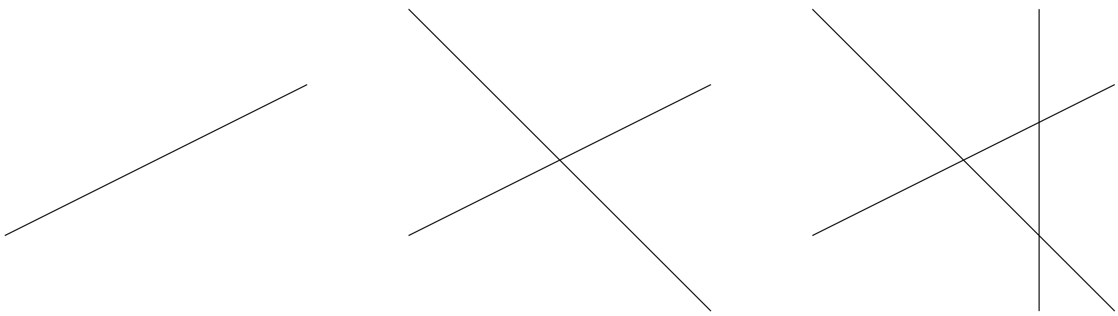
$$a_0 = 0 \quad \text{and} \quad a_n = 3a_{n-1} + 4 \quad \text{for } n \geq 1.$$

Prove by induction that $a_n = 2(3^n - 1)$ for all integers $n \geq 0$.

[*3 marks*]

$*$ Question 3

The diagrams below show a plane divided into regions by different numbers of straight lines. Suppose r_n is the maximum number of regions a plane can be divided into by n straight lines.



$r_1 = 2$ $r_2 = 4$ $r_3 = 7$

(a) Write down a recurrence relation expressing r_{n+1} in terms of r_n for $n \geq 0$, with an explanation justifying it.

(b) Use this to prove by induction that $r_n = \dfrac{n^2 + n + 2}{2}$.

[*4 marks*]

Solution 1

(a) The inequality $3^n < n!$ holds for positive integers $n \geq 7$.

That alone is sufficient to answer the question as given, but the following text gives some explanation of how you might work out that answer.

The following table records 3^n and $n!$ for positive integers n up to 8.

n	3^n	$n!$
1	3	1
2	9	2
3	27	6
4	81	24
5	243	120
6	729	720
7	2187	5040
8	$3 \cdot 2187$	$8 \cdot 5040$

This table shows that $3^n \not< n!$ for $n = 1, \ldots, 6$, that $3^n < n!$ for $n = 7$, and we observe that from then on $n!$ increases faster than 3^n.

That's not a bad proof on its own, but the second half of the question requires us to make this precise with a proof by induction.

(b) Define $P(n)$ as the statement $3^n < n!$. We show by induction that $P(n)$ holds for all integers $n \geq 7$.

The base case, $P(7)$, follows from the table above.

For the induction step, assume $P(k)$ for some $k \geq 7$. We have to show $P(k+1)$, that $3^{k+1} < (k+1)!$. Noting that $3 < 7 \leq k < k+1$ and $3^k < k!$ we can reason as follows:

$$3^{k+1} = 3 \cdot 3^k < (k{+}1) \cdot 3^k < (k{+}1) \cdot k! = (k{+}1)!.$$

By induction $P(n)$ holds for all $n \geq 7$.

Solution 2

For the base case $n = 0$ we check that $2(3^0 - 1) = 2(1 - 1) = 0 = a_0$ as required.

For the step case we assume as induction hypothesis that $a_k = 2(3^k - 1)$ for integer $k \geq 0$ and try to show that $a_{k+1} = 2(3^{(k+1)} - 1)$. We can start by expanding both sides of this equation.

$$\begin{aligned} a_{k+1} &= 3a_k + 4 && \text{by definition of sequence} \\ &= 3(2(3^k - 1)) + 4 && \text{by induction hypothesis} \\ &= 6 \cdot 3^k - 6 + 4 \\ &= 6 \cdot 3^k - 2 \end{aligned} \qquad \begin{aligned} 2(3^{(k+1)} - 1) &= 2 \cdot 3 \cdot 3^k - 2 \\ &= 6 \cdot 3^k - 2 \end{aligned}$$

Both sides are equal and we have shown $a_{k+1} = 2(3^{(k+1)} - 1)$ as required.

From the base and step case we deduce by mathematical induction that $a_n = 2(3^n - 1)$ for all integers $n \geq 0$.

* **Solution 3**

(a) Each new line crosses up to n existing lines: and by choosing it not parallel to any and avoiding existing intersections we can ensure it always has exactly n distinct crossings.

That means it passes through $(n+1)$ existing regions, dividing each into two and so adding $(n+1)$ new regions. This gives us the following recurrence relation:

$$r_{n+1} = r_n + (n+1)\,.$$

(b) For the base case $n=1$ we confirm that $\dfrac{1^2+1+2}{2} = 2 = r_1$ as required.

For the step case we assume as induction hypothesis that $r_k = \dfrac{k^2+k+2}{2}$ for an arbitrary integer $k \geq 1$ and try to show that $r_{k+1} = \dfrac{(k+1)^2+(k+1)+2}{2}$. We start by expanding both sides of this equation. First the left:

$$\begin{aligned} r_{k+1} &= r_k + (k+1) && \text{from part (a)} \\ &= \frac{k^2+k+2}{2} + (k+1) && \text{by induction hypothesis} \\ &= \frac{k^2+k+2}{2} + \frac{2k+2}{2} = \frac{k^2+3k+4}{2} \end{aligned}$$

and then the right:

$$\frac{(k+1)^2+(k+1)+2}{2} = \frac{(k^2+2k+1)+(k+1)+2}{2} = \frac{k^2+3k+4}{2}\,.$$

Both sides are equal as required.

From the base and step case we deduce by mathematical induction that $r_n = \dfrac{n^2+n+2}{2}$ for all integers $n \geq 1$.