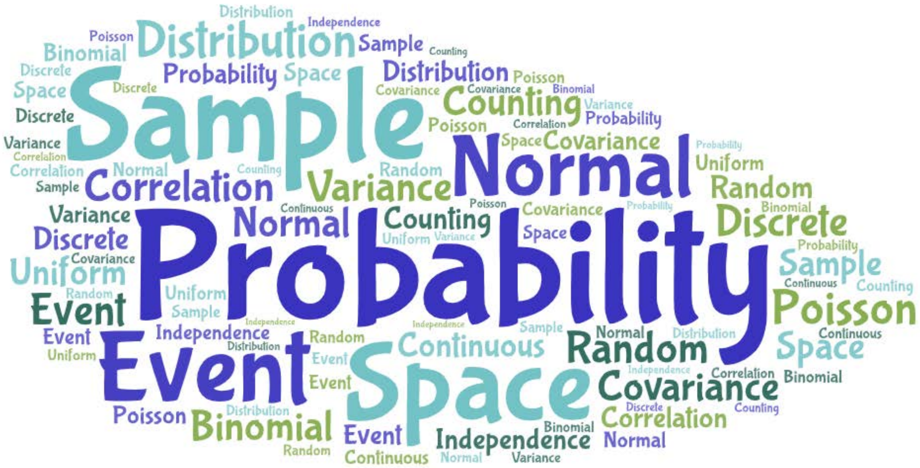
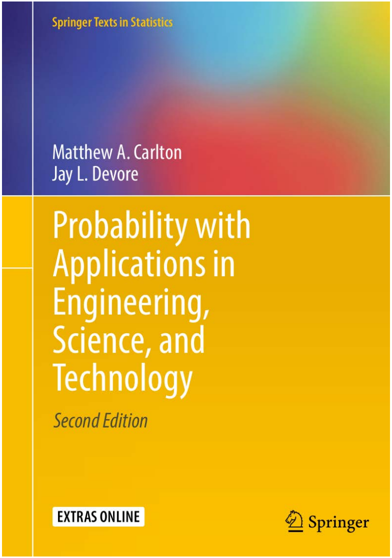
# Discrete Mathematics and Probability

Week 6




Heather Yorston


Springer Texts in Statistics
Matthew A. Carlton
Jay L. Devore
Probability with Applications in Engineering, Science, and Technology
Second Edition
EXTRAS ONLINE
Springer

# Topics

- Counting: thinking algorithmically
- Events: what could happen in principle
- Experiments: how can events interact
- Probability: quantifying what could happen

# Counting

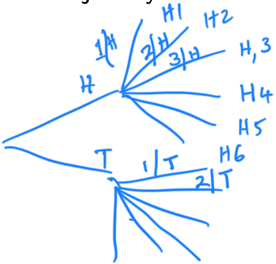
# Counting

Basic principles of combinatorics:

- if an experiment has $n$ outcomes;
  and another experiment has $m$ outcomes,
- then the two experiments jointly have $n \cdot m$ outcomes.

Coin

6 sided Die
fair




$2 \cdot 6 = 12$ outcomes.

# Permutations

## Definition

Let $H = \{h_1, h_2, \ldots, h_n\}$ be a set of $n$ different objects. The *permutations* of $H$ are the different orders in which you can write all of its elements.

Car Race $\{A, B, C, D, E\}$

A B C E D

$5 \times 4 \times 3 \times 2 \times 1 = 5! = 120$ permutations

$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots 1$

$0! = 1$

# Permutations with repetitions

## Definition

Let $H = \{h_1 \ldots h_1,\ h_2 \ldots h_2,\ \ldots,\ h_r \ldots h_r\}$ be a set of $r$ different types of repeated objects: $n_1$ many of $h_1$, $n_2$ of $h_2$, $\ldots$ $n_r$ of $h_r$. The *permutations with repetitions* of $H$ are the different orders in which you can write all of its elements.

$n = n_1 + n_2 + \cdots n_r$ total $n_r$ objects.

$$\frac{n!}{n_1!\, n_2! \ldots n_r!}$$

ABRACADABRA.

| A | B | C | D | R |
|---|---|---|---|---|
| 5 | 2 | 1 | 1 | 2 |

$$\frac{11!}{5!\, 2!\, 1!\, 1!\, 2!}$$

# $k$-Permutations

**Definition**

Let $H = \{h_1, h_2, \ldots, h_n\}$ be a set of $n$ different objects. The *$k$-permutations* of $H$ are the different ways in which one can pick and write $k$ of its elements of $H$ in order.

$$_nP_k = \frac{n!}{(n-k)!}$$

Top 3 in a car race with 5 cars $\{A, B, C, D, E\}$

$$_5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = 60$$

$ABC \neq CAB$

# $k$-Permutations with repetitions

**Definition**

Let $H = \{h_1 \ldots, h_2 \ldots, \ldots, h_r \ldots\}$ be a set of $r$ different types of repeated objects, each of infinite supply. The *$k$-permutations with repetitions* of $H$ are the different orders in which one can write an ordered sequence of length $k$ using the elements of $H$.

3 letter words English alphabet.

$= 26 \times 26 \times 26$

$= 26^3$

How many bit strings of length k.
$\{0,1\}$

$= 2^k$

# $k$-Combinations

k- Combinations

## Definition

Let $H = \{h_1, h_2, \ldots, h_n\}$ be a set of $n$ different objects. The *$k$-combinations* of $H$ are the different ways in which one can pick $k$ of its elements without order.

$C(n,k)$ $\quad {}_nC_k = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$ $\quad$ ABC

$C^n_k$ $\quad$ choose. $\quad$ ↑ unordered. $\quad$ = BAC = CAB.

Ex Ways of forming a committee of 5 students from a class of 30.

$${}_{30}C_5 = \binom{30}{5} = \frac{30!}{5!\,25!} = 142506$$

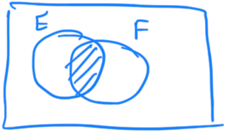
$${}_nP_k = \frac{n!}{(n-k)!}$$

# Events

# Events

A mathematical model for experiments:

- *Sample space*: the set $\Omega$ of all possible outcomes
- An *event* is a collection[1] of possible outcomes: $E \subseteq \Omega$
- *Union* $E \cup F$ and *intersection* $E \cap F$ of events make sense


[1]Sometimes $\Omega$ is too large, and not all subsets are events. Ignore this now.

# Examples

# Union and intersection

## Union

*Union* $E \cup F$ of events $E$ and $F$ means either $E$ or $F$ or both.
Infinite union $\bigcup_i E_i$ of events $E_i$ means at least one of the $E_i$'s.

## Intersection

*Intersection* $E \cap F$ of events $E$ and $F$ means both $E$ and $F$.
Infinite intersection $\bigcap_i E_i$ of events $E_i$ means each of the $E_i$'s.

## Definition

If $E \cap F = \emptyset$, we call events $E$ and $F$ *mutually exclusive*.
If events $E_1, E_2, \ldots$ satisfy $E_i \cap E_j = \emptyset$ whenever $i \neq j$, we call them *mutually exclusive*. They cannot happen at the same time.

# Inclusion and implication

**Remark**

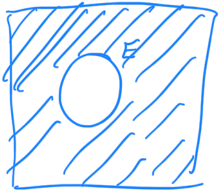
If the event $E$ is a *subset* of the event $F$, written $E \subseteq F$, then the occurrence of $E$ *implies* that of $F$.

# Complementarity

**Definition** $E^c$ or $\bar{E}$

The *complement* of an event $E$ is $E^c = \Omega - E$.

This is the event that $E$ does *not* occur.




$$E \cap E^c = \phi$$

$$E \cup E^c = \Omega$$

# Experiments

# Experiments

How events can interact:

- Commutativity
- Distributivity
- Associativity
- De Morgan's Law

# Properties of events

- Commutativity: $E \cup F = F \cup E$
  $E \cap F = F \cap E$
- Associativity: $E \cup (F \cup G) = (E \cup F) \cup G$
  $E \cap (F \cap G) = (E \cap F) \cap G$

# Properties of events

- Distributivity: $(E \cup F) \cap G = (E \cap G) \cup (F \cap G)$
  $(E \cap F) \cup G = (E \cup G) \cap (F \cup G)$

# De Morgan's law

- De Morgan's law: $(E \cup F)^c = E^c \cap F^c$

# Probability

- Definition by axioms
- How to compute probabilities
- Inclusion-exclusion principle
- Equally likely outcomes

# Axioms of probability

## Definition

The *probability* $\mathbf{P}$ on a sample space $\Omega$ assigns *numbers* to *events* of $\Omega$ in such a way that:

1. the probability of any event is non-negative: $\mathbf{P}(E) \geq 0$;
2. the probability of the sample space is one: $\mathbf{P}(\Omega) = 1$;
3. for countably many *mutually exclusive* events $E_1, E_2, \ldots$:

$$\mathbf{P}\Big(\bigcup_i E_i\Big) = \sum_i \mathbf{P}(E_i)$$

# How to compute probabilities

**Proposition**

*For any event,* $\mathbf{P}(E^c) = 1 - \mathbf{P}(E)$.

**Corollary**

*We have* $\mathbf{P}(\emptyset) = \mathbf{P}(\Omega^c) = 1 - \mathbf{P}(\Omega) = 1 - 1 = 0$.
*For any event,* $\mathbf{P}(E) = 1 - \mathbf{P}(E^c) \leq 1$.

# How to compute probabilities


**Proposition**

*For any two events,* $\mathbf{P}(E \cup F) = \mathbf{P}(E) + \mathbf{P}(F) - \mathbf{P}(E \cap F)$.

**Proposition (*Boole's inequality*)**

*For any events* $E_1, E_2, \ldots, E_n$*:*

$$\mathbf{P}\left(\bigcup_{i=1}^{n} E_i\right) \leq \sum_{i=1}^{n} \mathbf{P}(E_i).$$

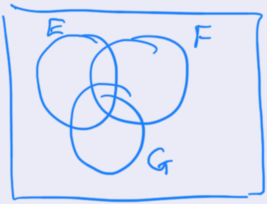
Proof by Induction

# Principle of Inclusion-exclusion (PIE)

## Proposition

*For any events:*

$$
\begin{aligned}
\mathbf{P}(E \cup F \cup G) = \mathbf{P}(E) + \mathbf{P}(F) + \mathbf{P}(G) \\
- \mathbf{P}(E \cap F) - \mathbf{P}(E \cap G) - \mathbf{P}(F \cap G) \\
+ \mathbf{P}(E \cap F \cap G).
\end{aligned}
$$



# Principle of Inclusion-exclusion (PIE)

## Proposition

*For any events:*

$$\begin{aligned}\mathbf{P}(E \cup F \cup G) = \mathbf{P}(E) + \mathbf{P}(F) + \mathbf{P}(G) \\ - \mathbf{P}(E \cap F) - \mathbf{P}(E \cap G) - \mathbf{P}(F \cap G) \\ + \mathbf{P}(E \cap F \cap G).\end{aligned}$$

$$\begin{aligned}\mathbf{P}(E_1 \cup E_2 \cup \cdots \cup E_n) &= \sum_{1 \le i \le n} \mathbf{P}(E_i) \\ &- \sum_{1 \le i_1 < i_2 \le n} \mathbf{P}(E_{i_1} \cap E_{i_2}) \\ &+ \sum_{1 \le i_1 < i_2 < i_3 \le n} \mathbf{P}(E_{i_1} \cap E_{i_2} \cap E_{i_3}) \\ &- \cdots \\ &+ (-1)^{n+1} \mathbf{P}(E_1 \cap E_2 \cap \cdots \cap E_n).\end{aligned}$$

# Example

## Example

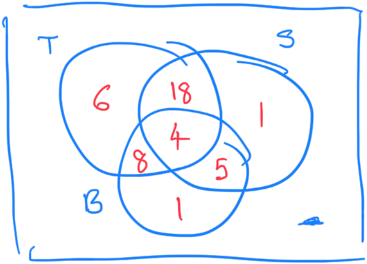
In a sports club,

36 members play tennis, 22 play tennis and squash,
28 play squash, 12 play tennis and badminton,
18 play badminton, 9 play squash and badminton,
4 play tennis, squash and badminton.

How many play at least one of these games?

$= 36+28+18 - 22 - 12 - 9 + 4$

$= 43$



# How to compute probabilities

**Proposition**

*If* $E \subseteq F$, *then* $\mathbf{P}(F - E) = \mathbf{P}(F) - \mathbf{P}(E)$.

**Corollary**

*If* $E \subseteq F$, *then* $\mathbf{P}(E) \leq \mathbf{P}(F)$.

# Equally likely outcomes

# The return of counting

Finite sample space, $|\Omega| = N < \infty$, has special important case where each experiment outcome has *equal probability*:

$$\mathbf{P}(\omega) = \frac{1}{N} \qquad \text{for all } \omega \in \Omega$$

**Definition**

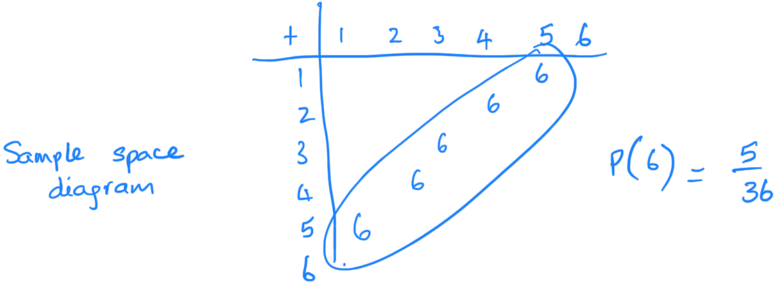
Outcomes $\omega \in \Omega$ are also called *elementary events*.

# Example

## Example

Rolling two dice, what is the probability that the sum of the numbers shown is 6?

What's wrong with this solution? "The number 6 is one out of the possible values $2, 3, \ldots, 12$ for the sum, and the answer is $\frac{1}{11}$."



# Summary

- Counting: permutations, combinations, repetitions
- Events: sample space, union, intersection, complement
- Experiments: distributivity, De Morgan's law
- Probability: axioms, how to compute, equally likely outcomes