Discrete Mathematics and Probability

Week 7

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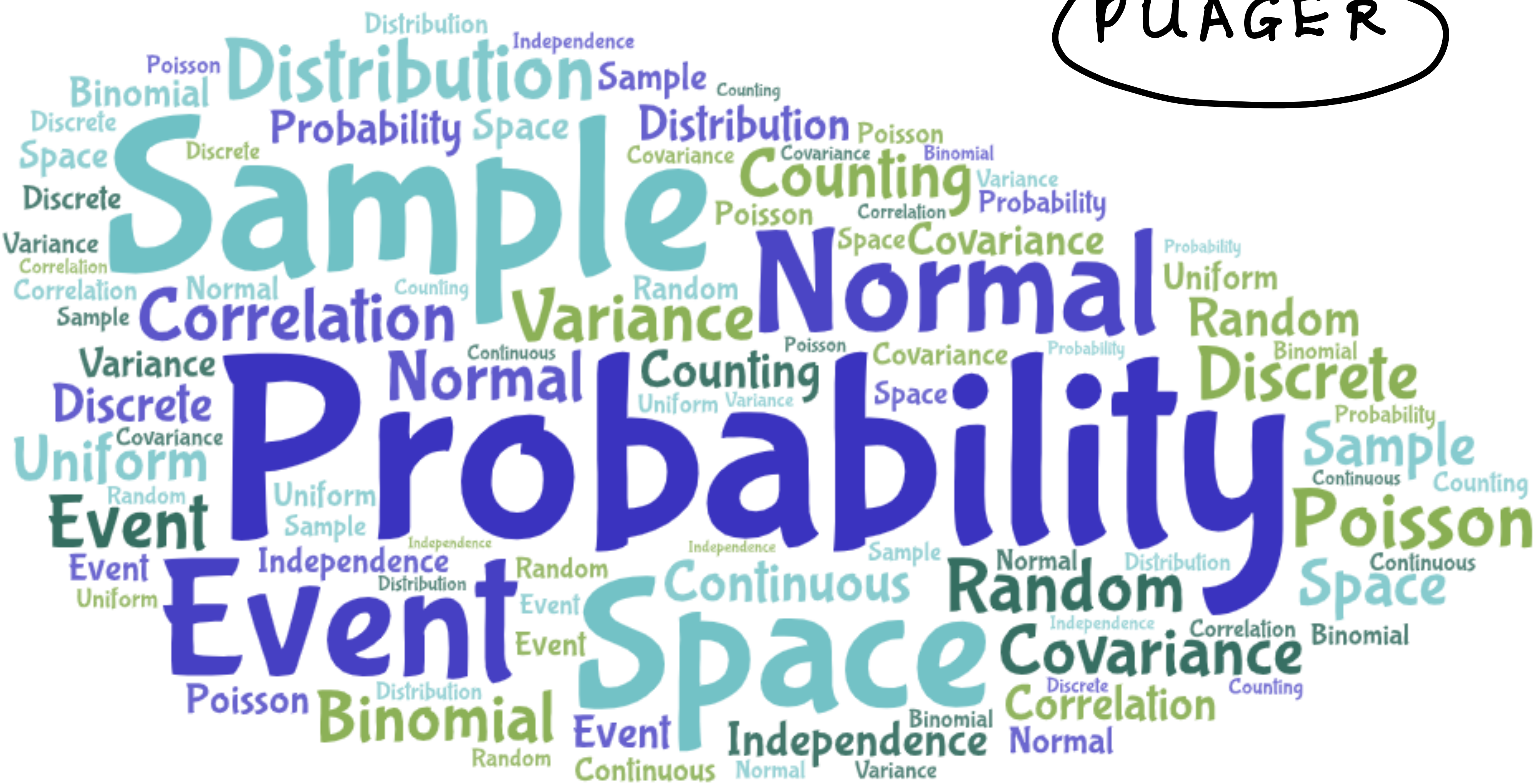

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Topics

- Bayes' Theorem example
- Independence: what information changes probability
- Random variables: when variables depend on chance
- Expectation: most likely outcomes of experiment
- Variance: how much the experiment can deviate

Bayes' Theorem Example

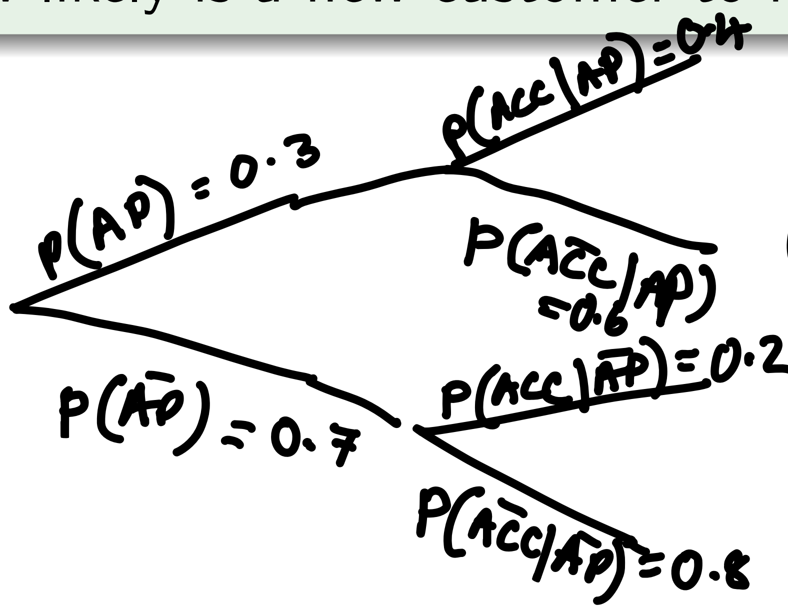
Example

Example

According to an insurance company:

- 30% of population are *accident-prone* (AP):
  they will have an accident (Acc) in any given year with 0.4 chance.
- 70% of population are *careful*:
  they have an accident in any given year with 0.2 chance.

How likely is a new customer to have an accident in 2024?



26%

$P(AP \cap Acc) = P(AP) \times P(Acc|AP)$
$= 0.3 \times 0.4 = 0.12$

$P(\overline{AP} \cap Acc) = P(\overline{AP}) \times P(Acc|\overline{AP})$
$= 0.7 \times 0.2$
$= 0.14$

$P(Acc) = P(\overline{AP} \cap Acc) + P(AP \cap Acc)$
$= 0.14 + 0.12 = 0.26$

Bayes' Theorem

Theorem (Bayes' Theorem)

$$\boxed{\mathbf{P}(F \mid E)} = \frac{\mathbf{P}(E \mid F) \cdot \mathbf{P}(F)}{\mathbf{P}(E \mid F) \cdot \mathbf{P}(F) + \mathbf{P}(E \mid F^c) \cdot \mathbf{P}(F^c)}$$

If $\{F_i\}_i$ *partitions* Ω, *then:*

$$\mathbf{P}(F_i \mid E) = \frac{\mathbf{P}(E \mid F_i) \cdot \mathbf{P}(F_i)}{\sum_j \mathbf{P}(E \mid F_j) \cdot \mathbf{P}(F_j)}$$

Belief update

Example

Consider the insurance company again. Imagine it's now 2025. We learn that the new customer did have an accident in 2024. Now what is the chance that they are accident-prone?

$$P(AP \cap Acc \mid Acc) = \frac{P(Acc \mid AP)\, P(AP)}{P(Acc)}$$

$$= \frac{0.4 \times 0.3}{0.26}$$

$$= \frac{0.12}{0.26} = \frac{6}{13} \approx 0.46$$

46%

Independence

Independence

$P(E \cap F) = P(E) * P(F \mid E)$ URN

Sometimes partial information on an experiment does not change the likelihood of an event.

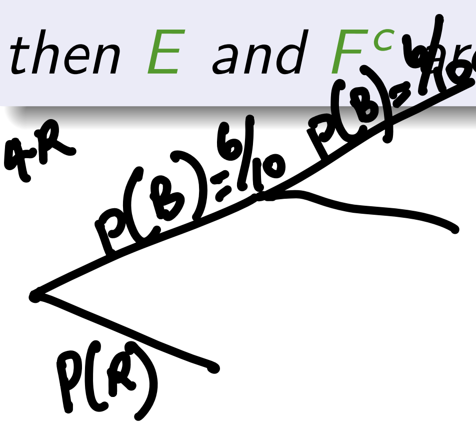
Definition

Events E and F are *independent* if $\mathbf{P}(E \mid F) = \mathbf{P}(E)$.
Equivalently: $\mathbf{P}(E \cap F) = \mathbf{P}(E) \cdot \mathbf{P}(F)$.
Equivalently: $\mathbf{P}(F \mid E) = \mathbf{P}(F)$.

Proposition

If E and F are independent events, then E and F^c are also independent.

6B 4R



with replacement

$P(B_2 \mid B_1) = \frac{6}{10} = P(B_2)$

Examples

+	1	2	3	4	5	6
1					6	7
2				6	7	
3			6	7		
4		6	7			
5	6	7				
6	7					

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① $S = \{1, 2, 3, \dots 6\}^2$

$E = \{\text{sum is } 6\}$

$F = \{1^{st} \text{ die is a } 3\}$

$\frac{1}{36} = p(E \cap F) \cancel{=}^{?} p(E)\,p(F)$

$\frac{5}{36} \cdot \frac{1}{6}$

not independent.

② $E' = \{\text{sum is } 7\}$

$F = \{1^{st} \text{ die is a } 3\}$

$G = \{2^{nd} \text{ die is } 4\}$

$\frac{1}{36} = p(E' \cap F) = p(E')\,p(F)$

$= \frac{\cancel{6}}{36} \cdot \frac{1}{\cancel{6}}$

independent.

$= \frac{1}{36}$ ✓

$\frac{1}{36} = p(E' \cap G) = \frac{6}{36} \cdot \frac{1}{6} = \frac{1}{36}$ ✓

$\frac{1}{36} = p(F \cap G) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$ ✓

pairwise independent

but $p(E' | F \cap G) =^{?} p(E)$

NOT MUTUALLY indep.t

or $\frac{1}{36} \Leftarrow p(E' \cap F \cap G) =^{?}$

$= p(E') \cdot p(F) \cdot p(G) = \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{6}{36} = \frac{1}{6}$

Independence

Definition

Three events E, F, G are *(mutually) independent* if:

$$\mathbf{P}(E \cap F) = \mathbf{P}(E) \cdot \mathbf{P}(F),$$
$$\mathbf{P}(E \cap G) = \mathbf{P}(E) \cdot \mathbf{P}(G),$$
$$\mathbf{P}(F \cap G) = \mathbf{P}(F) \cdot \mathbf{P}(G),$$
$$\underline{\mathbf{P}(E \cap F \cap G)} = \underline{\mathbf{P}(E) \cdot \mathbf{P}(F) \cdot \mathbf{P}(G)}. \quad \times$$

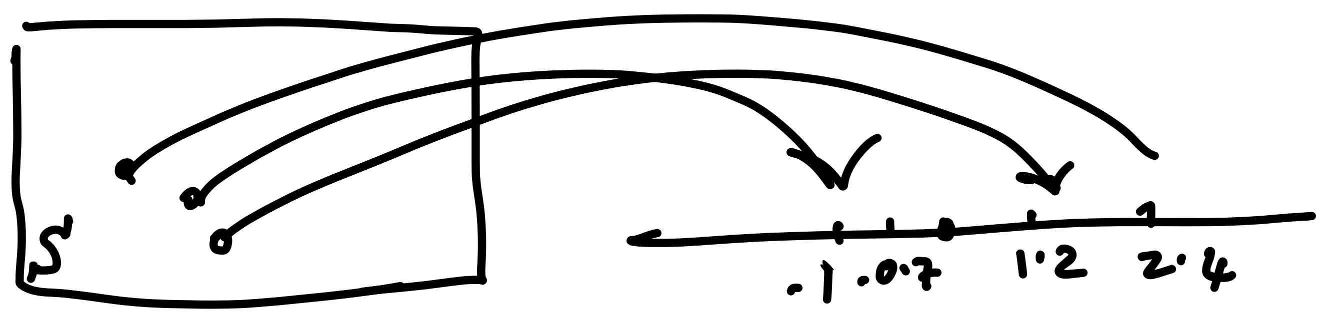
For more events the definition is that any (finite) subset of them have this factorisation property.

Random variables

Random variables

Definition

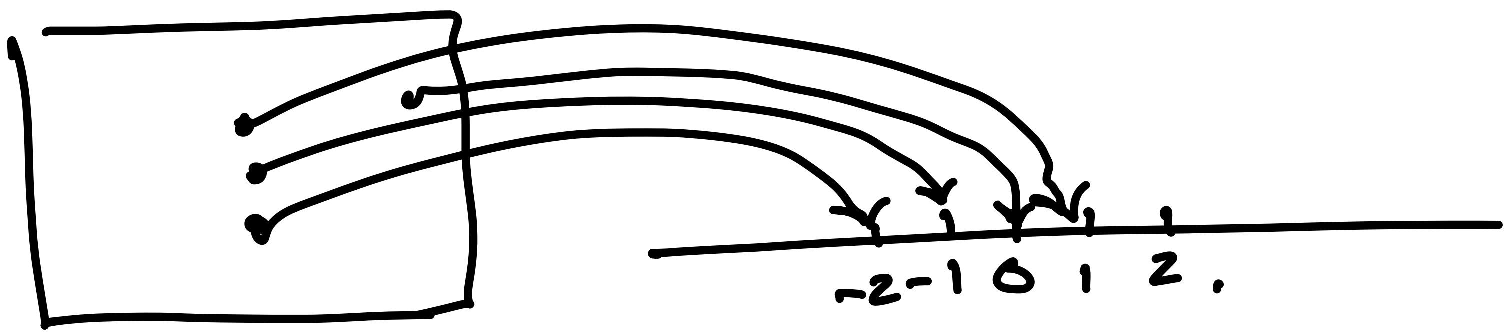
A *random variable* is a function from the sample space Ω to the real numbers $\mathbb{R}$.



Discrete random variables

Definition

A **discrete** random variable is an rv whose possible values constitute either a finite set or a countably infinite set (e.g., the set of all integers, or the set of all positive integers).



Probability mass function

Definition

The *probability mass function (pmf)*, or *distribution* of a discrete random variable X gives the probabilities of its possible values:

$$\mathfrak{p}_X(x_i) = \mathbf{P}(X = x_i),$$

Proposition

$$\mathfrak{p}(x_i) \geq 0 \qquad \textit{and} \qquad \sum_i \mathfrak{p}(x_i) = 1$$

Examples

X is the number of heads obtained on 3 coin flips

$D(X) = \{HHH\}$ $\quad p(HHH) = p(3) = \left(\frac{1}{2}\right)^3 = p(0)$

$p(HTT)$ or $p(THT)$ or $p(TTH)$

$p(1) = \frac{3}{8}$

$p(2) = \frac{3}{8}$

HHT
T
T

$Var(X) = E(X^2) - \{E(X)\}^2$
$= \underline{E(X^2)} - \mu^2$

$SD(X) = \sqrt{VAR(X)}$

x	0	1	2	3
$p(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Expectation or mean $\mu = E(X) = x\,P(X = x) = 0 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + 2\left(\frac{3}{8}\right) + 3\left(\frac{1}{8}\right) = \frac{12}{8}$

$Var(X) = x^2 P(X = x)$ $\quad Var(X) = 0^2 \frac{1}{8} + 1^2 \frac{3}{8} + 2^2\left(\frac{3}{8}\right) + 3^2\left(\frac{1}{8}\right) =$

Expectation

Expectation

Once we have a random variable, we want to quantify its *typical* behaviour in some sense. Two of the most often used quantities for this are the *expectation* and the *variance*.

Definition

The *expectation* of a discrete random variable X is:

$$\mathbf{E}(X) = \sum_i x_i \cdot \mathfrak{p}(x_i) = \mu$$

provided the sum exists. Also called *mean*, or *expected value*.

Moments

Definition (moments)

Let $n \in \mathbb{N}$. The n^{th} *moment* of a random variable X is:

$$\mathbf{E}(X^n)$$

The n^{th} *absolute moment* of X is:

$$\mathbf{E}(|X|^n)$$

Summary

- Independence: what information changes probability
- Random variables: when variables depend on chance
- Expectation: most likely outcomes of experiment
- Variance: how much the experiment can deviate