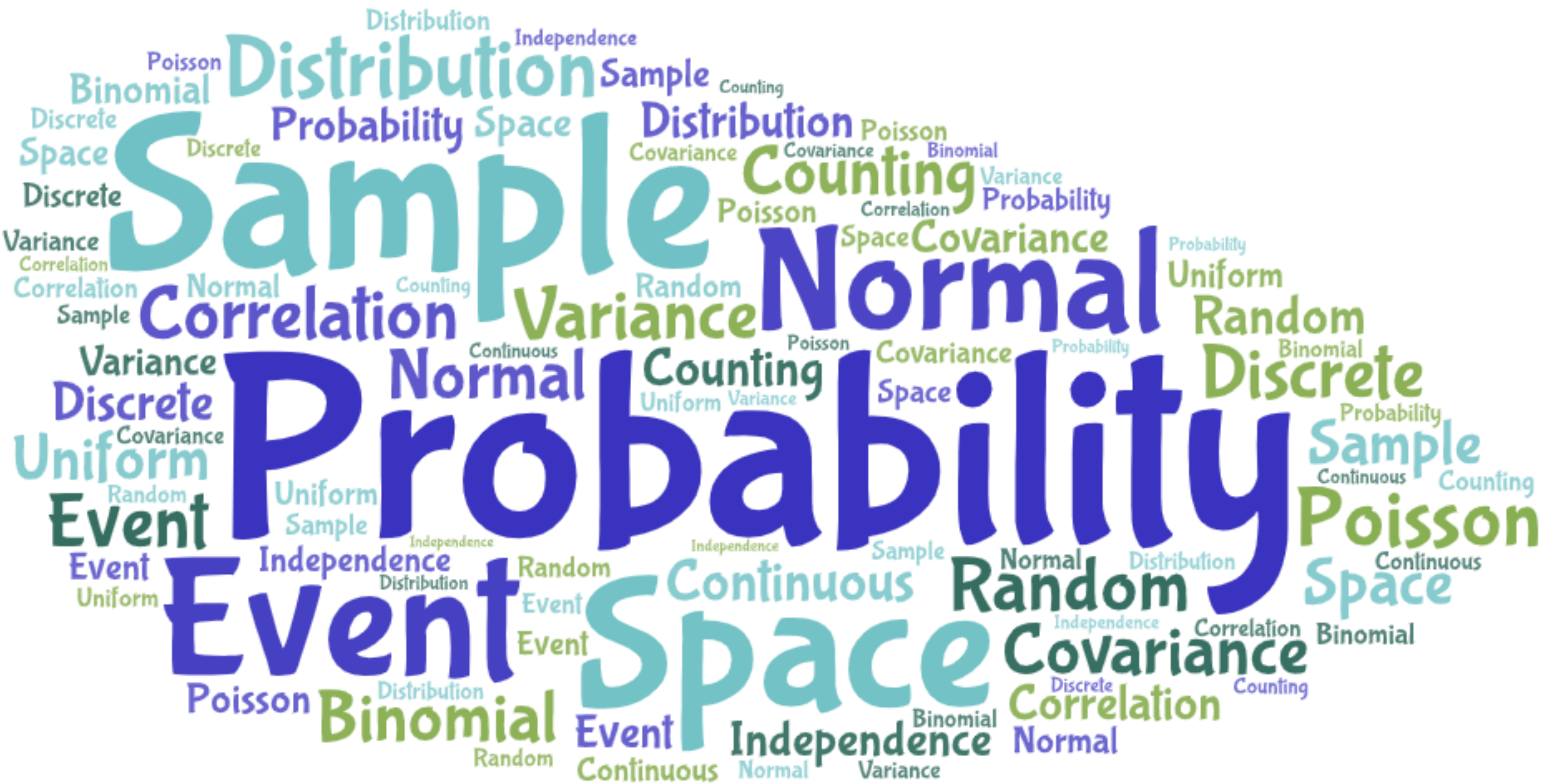
Discrete Mathematics and Probability

Week 8



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Topics

- Bernoulli distribution: single trial
- Binomial distribution: many independent trials
- Poisson distribution: counting independent trials
- Geometric distribution: first success in independent trials

Bernoulli and Binomial distributions

Bernoulli distribution

Definition

Suppose that n independent trials are performed, each succeeding with probability p. Let X count the number of successes within the n trials. Then X has the *Binomial distribution with parameters n and p* or, in short, $\underline{X \sim Bin(n, p)}$.

? $\sim$

Special case $n = 1$ is called *Bernoulli distribution with parameter p*.

Bernoulli: mass function

Proposition

Let $X \sim Bin(n, p)$. Then $X = 0, 1, \ldots, n$, and its mass function is

$$\boxed{\mathfrak{p}(i) = \mathbf{P}(X = i) = \binom{n}{i} p^i (1-p)^{n-i},} \qquad i = 0, 1, \ldots, n.$$

In particular, the $Bernoulli(p)$ variable can take on values 0 or 1, with respective probabilities

$$\mathfrak{p}(0) = 1 - p, \qquad \mathfrak{p}(1) = p.$$

$p(\text{not a six})$

$= 1 - \frac{1}{6} = \frac{5}{6}$

$p(\text{a six on 1 roll of a die}) = \frac{1}{6}.$

Mass function

$n = 1$ trial with 1 event

$P(X=1) = p = 1 - p(X=0) = 1 - p. \quad (q).$

$$f(k:p) \begin{cases} p & \text{if } k=1 \text{ (success)} \\ 1-p = q & \text{if } k=0 \text{ (failure)} \end{cases}$$

$$f(k:p) = p^{k} \cdot (1-p)^{1-k} \quad \text{for } k \in \{0,1\}$$

Bernoulli distribution is a special case of the Binomial distribution with $n=1$.

Bernoulli: expectation, variance

Proposition

Let $X \sim \mathrm{Ber}(p)$. Then:

$$\mathbf{E}(X) = p, \quad \text{and} \quad \mathbf{Var}(X) = p(1-p) = pq$$

Note $1 - p = q$

$X \sim \mathrm{Ber}(p)$

$E(X) = p.$ $\quad$ $\mathrm{Var}(X) = p(1-p)$

$\mu = E(X) = \sum x\, p(x) = 1\, P(X=1) + 0\, P(X=0)$

$= 1 \cdot p + 0 \cdot q = p$

$*$ $\mathrm{Var}(X) = E(X^2) - \mu^2 = p - p^2 = p(1-p) = pq.$

$E(X^2) = \sum x^2 p(x) = 1^2 p(X=1) + 0^2 P(X=0) = p$

Proof

Example

Decahedral die with faces numbered 1, 2, ... 10

X is obtaining a square number when the die is rolled once.

$S = \{1, 2, \ldots 10\}$

$E = \{1, 4, 9\}$

$$p(x) = \frac{|E|}{|S|} = \frac{3}{10} = 0.3$$

$X \sim \text{Ber}(0.3)$

x	0	1
$p(X = x)$	0.7	0.3.

$$\mu = E(X) = 0 \times 0.7 + 1 \times 0.3 = 0.3 = p.$$

$$\text{Var}(X) = pq = 0.3 \times (0.7) = 0.21$$

Proposition

Let $X \sim Bin(n, p)$. *Then* $X = 0, 1, \ldots, n$, *and its mass function is*

$$\mathfrak{p}(i) = \mathbf{P}(X = i) = \binom{n}{i} p^i (1-p)^{n-i}, \qquad i = 0, 1, \ldots, n.$$

Binomial variate can be considered as the sum of n independent Bernoulli variates with the probability of 'success'.

$$X \sim Bin(n,p) \equiv X \sim Ber(p) + Ber(p) + \ldots \ldots Ber(p) \text{ n times}$$

$$\mu_{BIN(n,p)} = E(X) = \mu_{Ber} + \mu_{Ber} + \ldots \mu_{Ber} \text{ n times}$$
$$= p + p + \ldots + p = np.$$

$$\sigma^2 \quad Var_{Bin(n,p)} = \underset{BER}{\sigma^2 VAR}(X) + \ldots \underset{BER}{Var}(X) \text{ (n times)}$$
$$= pq + \ldots + pq = npq$$

Examples

Probability of obtaining 2 sixes on tossing a unbiased die 3 times

X ~~is~~ obtained a six

N is not a six

S is a six

$$S = \{NNN, NNS, NSN, SNN, NSS, SNS, SSN, SSS\}.$$

$$E = \{\overset{\frac{5}{6}\frac{1}{6}\frac{1}{6}}{NSS}, SNS, SSN\}.$$

$$p(x = S) = \frac{1}{6}$$

$$p(x = N) = \frac{5}{6}.$$

$$X \sim Bin\left(3, \frac{1}{6}\right)$$

$$p(X = SS) = 3 \times \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)$$

Poisson distribution

Poisson: mass function

λ is mean number of events

The Poisson distribution is of central importance in Probability.
Will later see relation to Binomial.

Definition

Fix a positive real number λ. The random variable X is *Poisson distributed with parameter* λ, in short $X \sim Poi(\lambda)$, if it is non-negative integer valued, and its mass function is

$$\mathfrak{p}(i) = \mathbf{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}, \qquad i = 0, 1, 2, \ldots$$

For $X \sim Poi(\lambda)$, $\mathbf{E}(X) = \mathbf{Var}(X) = \lambda$.

Poisson approximation of Binomial

Proposition

Fix $\lambda > 0$, and suppose that $Y_n \sim Bin(n, p)$ with $p = p(n)$ in such a way that $n \cdot p \to \lambda$. Then the distribution of Y_n converges to $Poisson(\lambda)$:

$$\forall i \geq 0 \qquad \mathbf{P}(Y_n = i) \underset{n\to\infty}{\longrightarrow} e^{-\lambda}\frac{\lambda^i}{i!}.$$

Example

Poisson : many indpt small prob events. summing up to "a few" in expectation.

$\text{Poi}(X=i)$

$= \dfrac{e^{-\lambda}\lambda^{i}}{i!}$

$0! \equiv 1$

exs :

- no of typos on a page of a book.
- no of citizens $\geq$ 100 yrs old in a city
- no of calls per hour in a customer centre
- no of customers in P.O. today.

Book has a average of $\frac{1}{2}$ typos per page.
chance that the next page has $\geq$ 3 typos

$X \sim \text{Poi}(\lambda)$. $\lambda = 1/2$

$P(X \geq 3) = 1 - P(X \leq 2) = 1 - P(X=0) - P(X=1) - P(X=2)$

$= 1 - \dfrac{e^{-1/2}\left(\frac{1}{2}\right)^0}{0!} - \dfrac{e^{-1/2}\left(\frac{1}{2}\right)^1}{1!} - \dfrac{e^{-1/2}\left(\frac{1}{2}\right)^2}{2!}$

$= 1.4\%$

Example

A Geiger counter is used in recording radioactive events. Each radioactive event arriving at the counter shows as a number. The number of radioactive events recorded in a room is on average 2 every 5 seconds - background count.

a) Find the probability of exactly 3 events in 5 seconds

b) The probability of more than 3 events in 5 seconds.

$\boxed{\lambda = 2}$

$$Poi(X=i) = \frac{e^{-\lambda}\lambda^{i}}{i!}$$

$\lambda = 2$ $i = 3$

$$X \sim Poi(2)$$

a) $$Poi(X=3) = \frac{2^3 e^{-2}}{3!}$$

b) $$1 - Poi_2(0) - Poi_2(1) - Poi_2(2) - Poi_2(3)$$

$\boxed{\text{Clarke}}$

No. of $\boxed{\text{V-1 bombs}}$ in a square	$E(X)$	Observed.
0	226.74	229
1	211.39	211

Geometric distribution

Geometric: mass function

Again independent trials, but now ask: when is the first success?

Definition

Suppose that independent trials, each succeeding with probability p, are repeated until the first success. The total number X of trials made has the *Geometric*(p) distribution (in short, $X \sim Geo(p)$).

Proposition

X can take on positive integers, with probabilities
$\mathfrak{p}(i) = (1-p)^{i-1} \cdot p, \; i = 1, 2, \ldots$.

$\frac{5}{6} > \frac{5}{6} \times \frac{5}{6}$

first six on rolling a die on 3rd go

$$\overset{N}{\frac{5}{6}} \times \overset{N}{\frac{5}{6}} \times \overset{S}{\frac{1}{6}}$$

Geometric: mass function

Corollary

The Geometric random variable is (discrete) memoryless:

$$\mathbf{P}\{X \geq n + k \mid X > n\} = \mathbf{P}\{X \geq k\}$$

for every $k \geq 1$, $n \geq 0$.

Geometric: expectation, variance

Proposition

For a Geometric(p) random variable X:

$$\mathbf{E}(X) = \frac{1}{p} \qquad \mathbf{Var}(X) = \frac{1-p}{p^2}$$

Example

Summary

- Bernoulli distribution: single trial
- Binomial distribution: many independent trials
- Poisson distribution: counting independent trials
- Geometric distribution: first success in independent trials