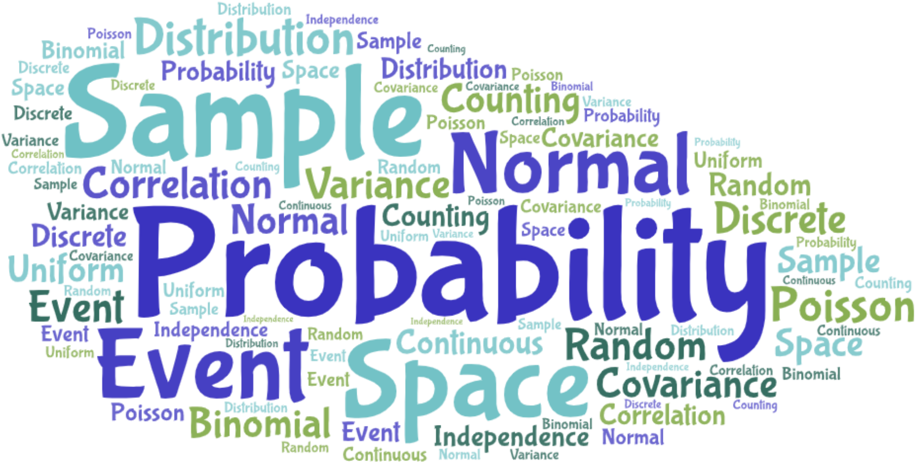
Discrete Mathematics and Probability

Week 10



Chris Heunen

Combinatorics and probability

- Counting: thinking algorithmically
  - Permutations $n!$, $n!/k!$
  - Combinations $\binom{n}{k} = n!/(k!(n-k)!)$
- Events: what could happen in principle
  - Sample space Ω
- Experiments: how can events interact
  - Complements, union, intersection
- Probability: quantifying what could happen
  - Axioms
  - Inclusion-exclusion
  - Equally likely outcomes

Axioms of probability

1. the probability of any event is non-negative: $\mathbf{P}(E) \geq 0$;
2. the probability of the sample space is one: $\mathbf{P}(\Omega) = 1$;
3. for countably many *mutually exclusive* events $E_1, E_2, \ldots$:

$$\mathbf{P}\left(\bigcup E_i\right) = \sum \mathbf{P}(E_i)$$

How to compute

- $\mathbf{P}(E^c) = 1 - \mathbf{P}(E)$
- $\mathbf{P}(E \cup F) = \mathbf{P}(E) + \mathbf{P}(F) - \mathbf{P}(E \cap F)$

Conditional probability

- Conditional probability: how knowledge influences probability
  - Partial information
  - How to compute: $\mathbf{P}(E \mid F) = \mathbf{P}(E \cap F)/\mathbf{P}(F)$
  - Reduced sample space
  - Axioms
  - How to compute: multiplication rule
- Bayes' theorem: link probabilities of related events
  - Partition theorem: $\mathbf{P}(E) = \sum_i \mathbf{P}(E \mid F_i) \cdot \mathbf{P}(F_i)$
  - Bayes' theorem:
    $\mathbf{P}(F_i \mid E) = \mathbf{P}(E \mid F_i) \cdot \mathbf{P}(F_i) / \sum \mathbf{P}(E \mid F_j) \cdot \mathbf{P}(F_j)$

Axioms of conditional probability

1. conditional probability is non-negative: $\mathbf{P}(E \mid) \geq 0$;
2. conditional probability of sample space is one: $\mathbf{P}(\Omega \mid F) = 1$;
3. for countably many *mutually exclusive* events $E_1, E_2, \ldots$:

$$\mathbf{P}\left(\bigcup E_i \,\middle|\, F\right) = \sum \mathbf{P}(E_i \mid F)$$

How to compute

- $\mathbf{P}(E^c \mid F) = 1 - \mathbf{P}(E \mid F)$
- $\mathbf{P}(E \mid F\} = 1 - \mathbf{P}(E^c \mid F) \leq 1$
- $\mathbf{P}(E \cup G \mid F) = \mathbf{P}(E \mid F) + \mathbf{P}(G \mid F) - \mathbf{P}(E \cap G \mid F)$
- If $E \subseteq G$, then $\mathbf{P}(G - E \mid F) = \mathbf{P}(G \mid F) - \mathbf{P}(E \mid F)$
- Multiplication rule: $\mathbf{P}(E_1 \cap \cdots \cap E_n) =$ $\mathbf{P}(E_1) \cdot \mathbf{P}(E_2 \mid E_1) \cdot \mathbf{P}(E_3 \mid E_1 \cap E_2) \cdots \mathbf{P}(E_n \mid E_1 \cap \cdots \cap E_{n-1})$

Random variables

- Independence: what information changes probability
  - Mutual independence: $\mathbf{P}(E \cap F) = \mathbf{P}(E) \cdot \mathbf{P}(F)$
- Random variables: when variables depend on chance
  - Probability mass function (pmf)
  - Cumulative distribution function (cdf)
- Expectation: most likely outcomes of experiment
  - Linear: $\mathbf{E}(aX + b) = a\mathbf{E}(X) + b$
  - How to compute: $\mathbf{E}(X) = \sum_i X(i)\mathbf{P}(i)$
- Variance: how much the experiment can deviate
  - How to compute: $\mathbf{Var}(X) = \mathbf{E}X^2 - (\mathbf{E}X)^2$
  - Not linear: $\mathbf{Var}(aX + b) = a^2 \cdot \mathbf{Var}(X)$
  - Standard deviation $\mathrm{SD}(X) = \sqrt{\mathbf{Var}\, X}$

Standard distributions

- Bernoulli distribution: single trial
  - Parameter p
- Binomial distribution: many independent trials
  - $\mathbf{P}(X = i) = \binom{n}{i} p^i (1-p)^{n-i}$
  - Expectation np, variance $np(1-p)$
- Poisson distribution: counting independent trials
  - Approximate binomial
  - Parameter λ
  - $\mathbf{P}(X = i) = e^{-\lambda}\lambda^i / i!$
  - Expectation and variance λ
- Geometric distribution: first success in independent trials
  - $\mathbf{P}(X = i) = (1-p)^{i-1} p$
  - Expectation $1/p$, variance $(1-p)/p^2$
  - Memoryless

Old exam question

Old exam question

Ask me anything!