Discrete Mathematics and Probability — **Induction and Recurrence**

Session 2024/25, Semester 1 — Week 3 Tutorial 2 with Solution Notes

Retrieve your submissions from Homework 1 in Week 2 as well as the solution notes on the course website. Compare solutions around the group.

What counterexample did you pick in Question 1(b)? Can you find others? Do all counterexamples n have $(n^2 + 4)$ a multiple of 5?

Now work together as a group on each of the following tasks.

Task A

Below are two constructions of numerical sequences. For each one carry out the following steps.

(a) Extend the sequence given for $n = 1, 2, 3, 4$, and 5.

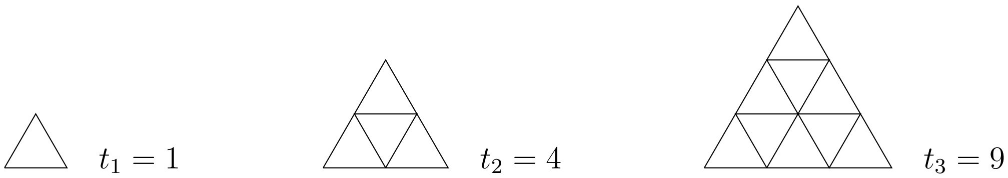
(b) Write down a conjecture for what you think the nth term of the sequence would be.

(c) Prove your conjecture correct or find a counterexample.

For both of these examples there is an obvious conjecture for how the sequence continues: one of those "obvious" conjectures is right and one is wrong.

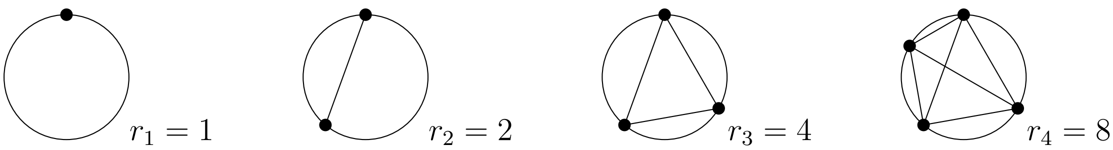
Triangulation

The sequence $t_1, t_2, \ldots, t_n, \ldots$ counts how many small triangles make a big triangle with n to a side, as shown below.



Circle Subdivision

The sequence $r_1, r_2, \ldots, r_n, \ldots$ counts the largest number of regions a circle can be divided into by straight lines joining n points on the circumference, as shown below.



Task B

Use mathematical induction to prove that for any integer $n \geq 0$, $7^n - 2^n$ is divisible by 5.

Task C

Suppose that $f_0, f_1, f_2, \ldots$ is a sequence defined as follows:

$$f_0 = 5 \qquad f_1 = 16 \qquad f_k = 7f_{(k-1)} - 10f_{(k-2)} \quad \text{for any integer } k \geq 2.$$

Prove by mathematical induction that $f_n = 3 \cdot 2^n + 2 \cdot 5^n$ for all non-negative integers n.

This is Question 6 from Exercise Set 5.4 in the Epp textbook. Does your solution use strong induction or not? How do you tell?

Task D

Sequence $c_0, c_1, c_2, \ldots$ is defined below.

$$c_0 = 2 \quad c_1 = 2 \quad c_2 = 6$$
$$c_k = 3c_{k-3} \quad \text{for every integer } k > 2.$$

Prove that all elements of the sequence are even.

Task E

Calculate 9^k for $k = 0, 1, 2, 3, 4$, and 5. Use these results to make a conjecture relating the parity of n to the units digit of 9^n for non-negative integers n.

Use mathematical induction to prove your conjecture.

Solution Notes

For Homework 1 see the solution notes on the course web pages.

For Question 1(b) there is an infinite supply of counterexamples: for example, if $n = (10k + 1)$ or $n = (10k - 1)$ for any integer k then $(n^2 + 4)$ is divisible by 5.

$$(10k \pm 1)^2 + 1 \;=\; 100k^2 \pm 20k + 1 + 4 \;=\; 5 \cdot (20k \pm 4k + 1)$$

Similar reasoning shows that $n = (26k \pm 3)$ then $(n^2 + 4)$ is a multiple of 13, giving infinitely more counterexamples not divisible by 5.

These infinities of counterexamples of all kinds are in fact to be expected: formulas that actually do generate primes are extremely hard to find and much more complex than $(n^2 + 4)$. See the Wikipedia page on a *Formula for Primes* to learn more about this.
https://en.wikipedia.org/wiki/Formula_for_primes

Task A

Triangulation

The first five values of the sequence are $1, 4, 9, 16$, and 25, suggesting $t_n = n^2$.

This turns out to be correct and can be proved in several ways. For example, by mathematical induction on the recurrence relation $t_{n+1} = t_n + n + (n + 1)$, noting that expanding the triangle of side n can be done by adding on the bottom two rows of triangles: one row of n pointing down and another of $(n + 1)$ pointing up. Or, by calculating the area of the large triangle as $\frac{\sqrt{3}}{4}n^2$ and of the small triangles as $\frac{\sqrt{3}}{4}$. Or even just remembering that area increases as the square of length.

Circle Subdivision

The first five values of the sequence are $1, 2, 4, 8$, and 16, suggesting $r_n = 2^{n-1}$.

This is incorrect: the next value $r_6 = 31$ as can be discovered by drawing out the diagram. The actual formula for r_n is a quartic polynomial:

$$r_n \;=\; \frac{n}{24}n^3 - 6n^2 + 23n - 18 + 1 \;=\; \binom{n}{4} + \binom{n}{2} + 1\,.$$

For more discussion, including proofs of this formula, see Wikipedia on *Dividing a Circle into Areas*.
https://en.wikipedia.org/wiki/Dividing_a_circle_into_areas.

Task B

This and the following notes are taken from the Instructor's Manual for the Epp textbook.

Proof (by mathematical induction): Let the property $P(n)$ be the sentence

$$7^n - 2^n \text{ is divisible by } 5. \qquad \leftarrow P(n)$$

We will prove that $P(n)$ is true for every integer $n \geq 0$.

Show that $\boldsymbol{P(0)}$ ***is true***: $P(0)$ is true because $7^0 - 2^0 = 1 - 1 = 0$ and 0 is divisible by 5 (since $0 = 5 \cdot 0$).

Show that for every integer $\boldsymbol{k \geq 0}$***, if*** $\boldsymbol{P(k)}$ ***is true then*** $\boldsymbol{P(k+1)}$ ***is true***:

Let k be any integer with $k \geq 0$, and suppose

$$7^k - 2^k \text{ is divisible by } 5. \qquad \leftarrow \begin{array}{l} P(k) \\ \text{inductive hypothesis} \end{array}$$

We must show that

$$7^{k+1} - 2^{k+1} \text{ is divisible by } 5. \qquad \leftarrow P(k+1)$$

By definition of divisibility, the inductive hypothesis is equivalent to the statement $7^k - 2^k = 5r$ for some integer r. Then

$$\begin{aligned}
7^{k+1} - 2^{k+1} &= 7 \cdot 7^k - 2 \cdot 2^k \\
&= (5+2) \cdot 7^k - 2 \cdot 2^k \\
&= 5 \cdot 7^k + 2 \cdot 7^k - 2 \cdot 2^k \\
&= 5 \cdot 7^k + 2(7^k - 2^k) && \text{by algebra} \\
&= 5 \cdot 7^k + 2 \cdot 5r && \text{by inductive hypothesis} \\
&= 5(7^k + 2r) && \text{by algebra.}
\end{aligned}$$

Now $7^k + 2r$ is an integer because products and sums of integers are integers. Therefore, by definition of divisibility, $7^{k+1} - 2^{k+1}$ is divisible by 5 *[as was to be shown]*.

Task C

Proof (by strong mathematical induction): Let the property $P(n)$ be the equation

$$f_n = 3\cdot 2^n + 2\cdot 5^n.$$

We will prove that $P(n)$ is true for every integer $n \geq 0$.

Show that $P(0)$ and $P(1)$ are true: By definition of $f_0, f_1, f_2, \ldots$, we have that $f_0 = 5$ and $f_1 = 16$. Since $3\cdot 2^0 + 2\cdot 5^0 = 3 + 2 = 5$ and $3\cdot 2^1 + 2\cdot 5^1 = 6 + 10 = 16$, both $P(0)$ and $P(1)$ are true.

Show that for every integer $k \geq 1$, if $P(i)$ is true for each integer i from 0 through k, then $P(k+1)$ is true: Let k be any integer with $k \geq 1$, and suppose

$$f_i = 3\cdot 2^i + 2\cdot 5^i \text{ for every integer } i \text{ with } 0 \leq i \leq k. \qquad \leftarrow \text{inductive hypothesis}$$

We must show that

$$f_{k+1} = 3\cdot 2^{k+1} + 2\cdot 5^{k+1}.$$

Now

$$\begin{aligned}
f_{k+1} &= 7f_k - 10f_{k-1} && \text{by definition of } f_0, f_1, f_2, \ldots \\
&= 7(3\cdot 2^k + 2\cdot 5^k) - 10(3\cdot 2^{k-1} + 2\cdot 5^{k-1}) && \text{by inductive hypothesis} \\
&= 7(6\cdot 2^{k-1} + 10\cdot 5^{k-1}) - 10(3\cdot 2^{k-1} + 2\cdot 5^{k-1}) && \text{since } 2^k = 2\cdot 2^{k-1} \text{ and } 5^k = 5\cdot 5^{k-1} \\
&= (42\cdot 2^{k-1} + 70\cdot 5^{k-1}) - (30\cdot 2^{k-1} + 20\cdot 5^{k-1}) \\
&= (42 - 30)\cdot 2^{k-1} + (70 - 20)\cdot 5^{k-1} \\
&= 12\cdot 2^{k-1} + 50\cdot 5^{k-1} \\
&= 3\cdot 2^2\cdot 2^{k-1} + 2\cdot 5^2\cdot 5^{k-1} \\
&= 3\cdot 2^{k+1} + 2\cdot 5^{k+1} && \text{by algebra,}
\end{aligned}$$

[as was to be shown].

[Since both the basis and the inductive steps have been proved, we conclude that $P(n)$ is true for every integer $n \geq 0$.]

Task D

Proof (by strong mathematical induction): Let the property $P(n)$ be the sentence

$$c_n \text{ is even.}$$

We will prove that $P(n)$ is true for every integer $n \geq 0$.

Show that $P(0)$, $P(1)$, and $P(2)$ are true: By definition of $c_0, c_1, c_2, \ldots$, we have that $c_0 = 2$, $c_1 = 2$, and $c_2 = 6$ and 2, 2, and 6 are all even. So $P(0)$, $P(1)$, and $P(2)$ are all true.

Show that for every integer $k \geq 2$, if $P(i)$ is true for each integer i from 0 through k, then $P(k+1)$ is true: Let k be any integer with $k \geq 2$, and suppose

$$c_i \text{ is even for every integer } i \text{ with } 0 \leq i \leq k \qquad \leftarrow \text{inductive hypothesis}$$

We must show that

$$c_{k+1} \text{ is even.}$$

Now by definition of $c_0, c_1, c_2, \ldots, c_{k+1} = 3c_{k-2}$. Since $k \geq 2$, we have that $0 \leq k-2 \leq k$, and so, by inductive hypothesis, c_{k-2} is even. Now the product of an even integer with any integer is even *[properties 1 and 4 of Example 4.2.3]*, and hence $3c_{k-2}$, which equals c_{k+1}, is also even *[as was to be shown]*.

[Since both the basis and the inductive steps have been proved, we conclude that $P(n)$ is true for every integer $n \geq 0$.]

Task E

$9^1 = 9,\ 9^2 = 81,\ 9^3 = 729,\ 9^4 = 6561,$ and $9^5 = 59049$.

Conjecture: For every integer $n \geq 0$, the units digit of 9^n is 1 if n is even and is 9 if n is odd.

Proof (by mathematical induction): Let the property $P(n)$ be the sentence

The units digit of 9^n is 1 if n is even and is 9 if n is odd.

We will prove that $P(n)$ is true for every integer $n \geq 0$.

Show that $P(0)$ is true: $P(0)$ is true because 0 is even and the units digit of $9^0 = 1$.

Show that for every integer $k \geq 0$, if $P(i)$ is true for each integer i from 0 through k, then $P(k+1)$ is true: Let k be any integer with $k \geq 0$, and suppose:

For every integer i from 0 through k,

$$\text{the units digit of } 9^i = \begin{cases} 1 \text{ if } i \text{ is even} \\ 9 \text{ if } i \text{ is odd} \end{cases} \quad \leftarrow \quad \text{inductive hypothesis}$$

We must show that

$$\text{the units digit of } 9^{k+1} = \begin{cases} 1 \text{ if } i \text{ is even} \\ 9 \text{ if } i \text{ is odd} \end{cases} \quad \leftarrow \quad P(k+1)$$

Case 1 (k+1 is even): In this case k is odd, and so, by inductive hypothesis, the units digit of 9^k is 9. This implies that there is an integer a so that $9^k = 10a + 9$, and hence

$$\begin{aligned} 9^{k+1} &= 9^1 \cdot 9^k && \text{by algebra (a law of exponents)} \\ &= 9(10a+9) && \text{by substitution} \\ &= 90a + 81 \\ &= 90a + 80 + 1 \\ &= 10(9a+8) + 1 && \text{by algebra.} \end{aligned}$$

Because $9a + 8$ is an integer, it follows that the units digit of 9^{k+1} is 1.

Case 2 ($k+1$ is odd): In this case k is even, and so, by inductive hypothesis, the units digit of 9^{k-1} is 1. This implies that there is an integer a so that $9^k = 10a + 1$, and hence

$$\begin{aligned} 9^{k+1} &= 9^1 \cdot 9^k && \text{by algebra (a law of exponents)} \\ &= 9(10a+1) && \text{by substitution} \\ &= 90a + 9 \\ &= 10(9a) + 9 && \text{by algebra.} \end{aligned}$$

Because $9a$ is an integer, it follows that the units digit of 9^{k+1} is 9.

Hence in both cases the units digit of 9^{k+1} is as specified in $P(k+1)$ *[as was to be shown]*.