Retrieve your submissions from Homework 2 in Week 3 as well as the solution notes on the course website. Compare solutions around the group.

How did you identify the (infinite) set of integers required for Question 1(a)?

For Question 3, what constraints do you need to put on placement of the n straight lines to make sure that the number of regions r_n is as large as possible? What is the *smallest* number of regions s_n that a plane can be divided into by n different straight lines?

Now work together as a group on each of the following tasks, some of which are taken from questions in the Epp textbook.

Task A

Find the mistake in the following "proof" that for all sets A, B, and C, if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

"**Proof:** Suppose A, B, and C are any sets such that $A \subseteq B$ and $B \subseteq C$. Since $A \subseteq B$, there is an element x such that $x \in A$ and $x \in B$, and since $B \subseteq C$, there is an element x such that $x \in B$ and $x \in C$. Hence there is an element x such that $x \in A$ and $x \in C$ and so $A \subseteq C$."

Task B

Find the mistake in the following "proof."

"Theorem:" For all sets A and B, $A^c \cup B^c \subseteq (A \cup B)^c$.

"**Proof:** Suppose A and B are any sets, and $x \in A^c \cup B^c$. Then $x \in A^c$ or $x \in B^c$ by definition of union. It follows that $x \notin A$ or $x \notin B$ by definition of complement, and so $x \notin A \cup B$ by definition of union. Thus $x \in (A \cup B)^c$ by definition of complement, and hence $A^c \cup B^c \subseteq (A \cup B)^c$."

Task C

Recall that for sets A and B if there is a bijection $f : A \to B$, a function that is both injective (one-to-one) and surjective (onto), then $|A| = |B|$. Let $E = \{0, 2, 4, \ldots\}$ be the set of non-negative even integers.

(a) Give an example of a function $g : E \to E$ that is injective but not surjective.

(b) Prove that $|\mathbb{Z}| = |E|$ by defining explicitly a bijection h from $\mathbb{Z}$ to E.

Task D

(a) For each of the following functions $f : (\mathbb{Z} \times \mathbb{Z}) \to \mathbb{Z}$ determine whether or not it is surjective.

- **(i)** $f(m, n) = m^2 + n^2$
- **(ii)** $f(m, n) = m$
- **(iii)** $f(m, n) = |n|$
- **(iv)** $f(m, n) = m - n$

(b) Given any two functions $g : A \to B$ and $f : B \to C$ prove or disprove each of the following statements.

- **(i)** If $f \circ g$ and g are injective then f is injective.
- **(ii)** If $f \circ g$ and f are injective then g is injective.

Task E

Below is a table of set identities from Chapter 6 of Epp. Use these to simplify the following set expression, for arbitrary A and B.

$$(A - (A \cap B)) \cap (B - (A \cap B))$$

Theorem 6.2.2 Set Identities

Let all sets referred to below be subsets of a universal set U.

1. *Commutative Laws:* For all sets A and B,

   (a) $A \cup B = B \cup A$ and (b) $A \cap B = B \cap A$.

2. *Associative Laws:* For all sets A, B, and C,

   (a) $(A \cup B) \cup C = A \cup (B \cup C)$ and
   (b) $(A \cap B) \cap C = A \cap (B \cap C)$.

3. *Distributive Laws:* For all sets A, B, and C,

   (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and
   (b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

4. *Identity Laws:* For every set A,

   (a) $A \cup \emptyset = A$ and (b) $A \cap U = A$.

5. *Complement Laws:* For every set A,

   (a) $A \cup A^c = U$ and (b) $A \cap A^c = \emptyset$.

6. *Double Complement Law:* For every set A,

   $(A^c)^c = A$.

7. *Idempotent Laws:* For every set A,

   (a) $A \cup A = A$ and (b) $A \cap A = A$.

8. *Universal Bound Laws:* For every set A,

   (a) $A \cup U = U$ and (b) $A \cap \emptyset = \emptyset$.

9. *De Morgan's Laws:* For all sets A and B,

   (a) $(A \cup B)^c = A^c \cap B^c$ and (b) $(A \cap B)^c = A^c \cup B^c$.

10. *Absorption Laws:* For all sets A and B,

    (a) $A \cup (A \cap B) = A$ and (b) $A \cap (A \cup B) = A$.

11. *Complements of U and $\emptyset$:*

    (a) $U^c = \emptyset$ and (b) $\emptyset^c = U$.

12. *Set Difference Law:* For all sets A and B,

    $A - B = A \cap B^c$.

Solution Notes

For Homework 2 see the solution notes on the course web pages.

The constraints on the lines are that none should be parallel and no three should pass through the same point: each intersection has to avoid every other one. The smallest number of regions s_n is $(n+1)$, obtained when all n lines are parallel. The other extreme situation, with all lines intersecting in a single point, gives $2n$ regions, and $2n \geq (n+1)$ for all positive integers n. Putting all lines on top of each other gives only 2 regions, but then they wouldn't be different lines. (If you did identify that case, though, congratulations on your skill in spotting awkward corner cases.)

Task A

This and other boxed notes are taken from the Instructor's Manual for the Epp textbook.

> There is more than one error in the "proof." The most serious is the misuse of the definition of subset. To say that A is a subset of B means that for every x, **if** $x \in A$ **then** $x \in B$. It does not mean that there exists an element of A that is also an element of B. The second error in the proof occurs in the last sentence. Even if there is an element in A that is in B and an element in B that is in C, it does not follow that there is an element in A that is in C. For instance, suppose $A = \{1,2\}$, $B = \{2,3\}$, and $C = \{3,4\}$. Then there is an element in A that is in B (namely 2) and there is an element in B that is in C (namely, 3), but there is no element in A that is in C.

Task B

> The "proof" claims that because $x \notin A$ or $x \notin B$, it follows that $x \notin A \cup B$. But it is possible for "$x \notin A$ or $x \notin B$" to be true and "$x \notin A \cup B$" to be false. For example, let $A = \{1,2\}$, $B = \{2,3\}$, and $x = 3$. Then since $3 \notin \{1,2\}$, the statement "$x \notin A$ or $x \notin B$" is true. But since $A \cup B = \{1,2,3\}$ and $3 \in \{1,2,3\}$, the statement "$x \notin A \cup B$" is false.

Task C

(a) There are several possible maps: for example $g(x) = 2x$, $g(x) = x^3$, or $g(x) = 7x$.

(b) The tricky part here is to define a function that handles both negative and positive numbers while making sure to hit every value of E. Here is one example.

$$h(x) \;=\; \begin{cases} 4x & \text{if } x \geq 0 \\ -4x-2 & \text{if } x < 0 \end{cases}$$

Task D

(a) For each example we need either to show a value in $\mathbb{Z}$ that is not in the image of f, or for arbitrary $z \in \mathbb{Z}$ find a pair in $\mathbb{Z} \times \mathbb{Z}$ that maps to z under f.

- **(i)** This function is not surjective: no negative integer is the sum of two squares, nor are some positive integers like 3 or 6.
- **(ii)** This function is surjective: for any $z \in \mathbb{Z}$ we have $f(z, x) = z$ for any $x \in \mathbb{Z}$.
- **(iii)** This function is not surjective: all values $f(x)$ are positive so there is no $z \in \mathbb{Z}$ with $f(z) = -1$, for example.
- **(iv)** This function is surjective: for any $z \in \mathbb{Z}$ we have $f(z, 0) = z$; indeed, for any $k \in \mathbb{Z}$ we have $f(z + k, k) = z$.

(b)

- **(i)** This statement is false: functions $f \circ g$ and g may be injective while f is not injective. For example, if $A = \{1, 2\}$, $B = \{1, 2, 3\}$, and $C = \{1, 2\}$ with $g(1) = 1$, $g(2) = 2$, $f(1) = 1$, $f(2) = 2$, and $f(3) = 1$. Then g is injective, $f \circ g$ is the identity map on $\{1, 2\}$ and so also injective, while $f(1) = f(3)$ so f is not injective.
- **(ii)** This statement is true. We can in fact prove a stronger statement: if $f \circ g$ is injective then g is injective, regardless of f. Suppose $a, a' \in A$ and $g(a) = g(a')$. Then we can reason as follows.

$$\begin{aligned}
(f \circ g)(a) &= f(g(a)) && \text{by definition of } (f \circ g) \\
&= f(g(a')) && \text{as } g(a) = g(a') \\
&= (f \circ g)(a') && \text{by definition of } (f \circ g)
\end{aligned}$$

Given that $f \circ g$ is injective then this tells us $a = a'$; and hence g is also injective.

Task E

$(A - (A \cap B)) \cap (B - (A \cap B))$

$$\begin{aligned}
&= (A \cap (A \cap B)^c) \cap (B \cap (A \cap B)^c) && \text{by the set difference law (used twice)} \\
&= A \cap ((A \cap B)^c \cap (B \cap (A \cap B)^c)) && \text{by the associative law for } \cap \\
&= A \cap (((A \cap B)^c \cap B) \cap (A \cap B)^c) && \text{by the associative law for } \cap \\
&= A \cap ((B \cap (A \cap B)^c) \cap (A \cap B)^c) && \text{by the commutative law for } \cap \\
&= A \cap (B \cap ((A \cap B)^c \cap (A \cap B)^c)) && \text{by the associative law for } \cap \\
&= A \cap (B \cap (A \cap B)^c) && \text{by the idempotent law for } \cap \\
&= (A \cap B) \cap (A \cap B)^c && \text{by the associative law for } \cap \\
&= \emptyset && \text{by the complement law for } \cap.
\end{aligned}$$