Task A

You decide to have a dinner party. Even though you are incredibly popular and have 14 different friends, you only have enough chairs to invite 6 of them.

(a) How many choices do you have for which 6 friends to invite?

(b) What if you need to decide not only which friends to invite but also where to seat them along your long table? How many choices do you have then?

(c) Discuss how your answers to the first two parts relate to each other and whether this makes good sense.

(d) How many ways could your group sit at your tutorial table?

You can leave factorials in your answers.

Task B

Seven women and nine men are in an Institute in the Informatics department at a university.

(a) How many ways are there to select a committee of five members from the Institute if at least one woman must be on the committee?

(b) How many ways are there to select a committee of five members from the Institute if at least one woman and at least one man must be on the committee?

Task C

A telegraph sends out three symbols on the communication line.

(a) Using 1 for symbol received and 0 for a missed symbol. List all the possible outcomes for the three symbols as a string of 3 bits.

(b) Represent the following events in a single Venn diagram:

$$\begin{aligned}
A_1 &= \{\text{only the first symbol is received}\}\\
A_2 &= \{\text{at least one symbol is received}\}\\
A_3 &= \{\text{exactly two symbols are received}\}\\
A_4 &= \{\text{less than two symbols are received}\}\\
A_5 &= \{\text{exactly one symbol is received}\}
\end{aligned}$$

NB: You may find it helpful to change the binary string into an integer for your Venn diagram.

Task D

Five cards are numbered as 1, 2, 3, 4, and 5. Three cards are randomly selected from the set and are lined up next to each other to form a 3-digit number x. Find the probabilities of the following events:

- **(a)** $A = \{x = 123\}$
- **(b)** $B = \{x \text{ does not contain the digit } 4\}$
- **(c)** $C = \{x \text{ is even}\}$
- **(d)** $D = \{x \text{ contains at least one of the digits } 1,\ 2\}$

Task E

- **(a)** How many functions $f\colon\{1,2,3,4,5\} \to \{a,b,c,d\}$ are there?
- **(b)** How many functions $f\colon\{1,2,3,4,5,6,7,8\} \to \{1,2,3,4,5,6,7,8\}$ are bijective?
- **(c)** How many functions $f\colon\{1,2,3\} \to \{1,2,3,4,5,6,7,8\}$ are injective?

Task F

In how many ways can you order the elements of the set $\{1, 2, \ldots, 2n\}$ so that every even number is at an even position?

Solution Notes

Task A

(a) You need to choose 6 friends from a group of 14 so that is ${}_{14}C_6 = \frac{14!}{8!6!} = 3003$ ways.

(b) Here you must count all the ways you can permute 6 friends chosen from a group of 14, that is ${}_{14}P_6 = \frac{14!}{8!} = 2162160$.

(c) In both counting problems we choose 6 out of 14 friends. For the first one, we stop there, at 3003 ways. But for the second counting problem, each of those 3003 choices of 6 friends can be arranged in exactly 6! ways. So now we have $6! \times 3003 = 2162160$ choices and that is exactly ${}_{14}P_6$.

(d) The answer is $n!$ where n is the number of people at your table. If there are more seats than people, though, you might count additional combinations where people sit in the same order around the table but in different seats: n people sitting in $m \geq n$ seats can be done in ${}_mP_n = \frac{m!}{(m-n)!}$ ways.

Task B

(a) The total number of ways to select a committee with 5 members of the department is $\binom{16}{5}$. Of those, $\binom{9}{5}$ have no women. Thus, those with at least one woman are $\binom{16}{5} - \binom{9}{5} = 4242$.

Alternatively, we can divide in four cases and count separately the choices with one woman $\binom{7}{1}\binom{9}{4}$, with two women $\binom{7}{2}\binom{9}{3}$, with three $\binom{7}{3}\binom{9}{2}$, with four $\binom{7}{4}\binom{9}{1}$ and with five $\binom{7}{5}$. Thus, the answer is

$$\binom{7}{1}\binom{9}{4} + \binom{7}{2}\binom{9}{3} + \binom{7}{3}\binom{9}{2} + \binom{7}{4}\binom{9}{1} + \binom{7}{5} = 4242.$$

(b) Applying a similar reasoning the answer is $\binom{16}{5} - \binom{9}{5} - \binom{7}{5}$, or

$$\binom{7}{1}\binom{9}{4} + \binom{7}{2}\binom{9}{3} + \binom{7}{3}\binom{9}{2} + \binom{7}{4}\binom{9}{1} = 4221.$$

Task C

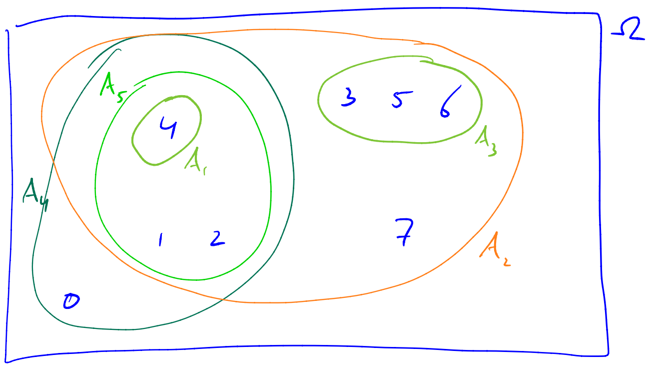
Write 0 and 1 for whether a symbol is received. The sample space consists of strings of three bits, or, changing them from binary notation to base 10 integers, of the numbers $\Omega = \{0, 1, 2, 3, 4, 5, 6, 7\}$.

000	001	010	011	100	101	110	111
0	1	2	3	4	5	6	7

The events become:

$$A_1 = \{4\}$$
$$A_2 = \{1, 2, 3, 4, 5, 6, 7\}$$
$$A_3 = \{3, 5, 6\}$$
$$A_4 = \{0, 1, 2, 4\}$$
$$A_5 = \{1, 2, 4\}$$

In a Venn diagram:



Task D

The number of possible permutations of 3 elements from a set of 5 is ${}_5P_3 = 5 \cdot 4 \cdot 3 = 60$.

(a) Only the permutation 123 is selected, so the event A has cardinality 1, and thus probability $P(A) = \frac{1}{60}$.

(b) To not contain the digit 4, we can choose only among digits 1, 2, 3, 5, for which there are ${}_4P_3 = 4 \cdot 3 \cdot 2 = 24$ possibilities, so $P(B) = \frac{24}{60} = \frac{2}{5}$.

(c) For x to be even, its last digit must be 2 or 4. In each of those cases, the first two digits can have one of 4 values (1,3,4,5 in the first case, and 1,2,3,5 in the second case). Therefore event C has cardinality $2 \cdot {}_4P_2 = 2 \cdot 4 \cdot 3 = 24$, and $P(C) = \frac{24}{60} = \frac{2}{5}$.

(d) Consider the complement $D^c = \{x \text{ does not contain the digits 1 or 2}\}$. Then every digit of x can be one of three values 3, 4, or 5, and so D^c has cardinality ${}_3P_3 = 3! = 6$, and so $P(D^c) = \frac{6}{60} = \frac{1}{10}$, and therefore $P(D) = 1 - P(D^c) = 1 - \frac{1}{10} = \frac{9}{10}$.

Task E

(a) 4^5

(b) $8!$

(c) ${}_8P_3$

Task F

There are n even numbers in the set. There are also n even positions in a sequence of $2n$ numbers. The number of ways to order n numbers in n positions is ${}_nP_n = n!$. Clearly, there are also n odd numbers in the set and they can also be ordered in n odd positions in $n!$ possible ways. So for each ordering of even numbers we have $n!$ orderings of odd numbers. Therefore the total number of orderings is $n! \cdot n! = (n!)^2$.