Task A

Discuss the difference between 'mutually exclusive' and 'independent' events.

You could find it helpful to explore this example.

Two dice are rolled and there are three events which we consider:

- A first die lands 1;
- B second die shows larger number than first die;
- C both dice show same number.

Which events are independent and which are mutually exclusive?

Task B

A random variable X has the following probability distribution:

x_i	1	3	4	6
p_i	0.25	0.3	0.3	0.15

Compute the cumulative distribution function (CDF) and from this calculate the probability $P(2 \leq X \leq 5)$.

Plot the probability mass function (PMF) and the cumulative distribution function.

Task C

An exam has 4 questions. Each question has 4 answers, of which exactly 1 is correct. The exam is given to 256 students. Each student answers each question randomly. What is the expected number of exam scripts with no correct answers? With one correct answer? With 2, with 3, with 4?

Consider the expected number of exam scripts with all four answers correct. What is the probability that the actual number of all-correct scripts is exactly that? What is the probability that it is lower? Higher? that?

Task D

Alice and Bob play take turns throwing a six-sided die. The first one to throw a 5 or 6 wins. Alice starts. What are the probability of the events $A = \{\text{Alice wins}\}$ and $B = \{\text{Bob wins}\}$?

(You may use the fact that $\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + x^8 + \cdots$.)

Solution Notes

Task A

$P(B|A) = \frac{5}{6}$ and $P(B) = \frac{15}{36}$ by counting from a table. These are not equal so A and B are **dependent**.

$P(C|A) = \frac{1}{6}$ and $P(C) = \frac{6}{36} = \frac{1}{6}$. These are identical so A and C are **independent**.

We note that B and C are **mutually exclusive**.

Task B

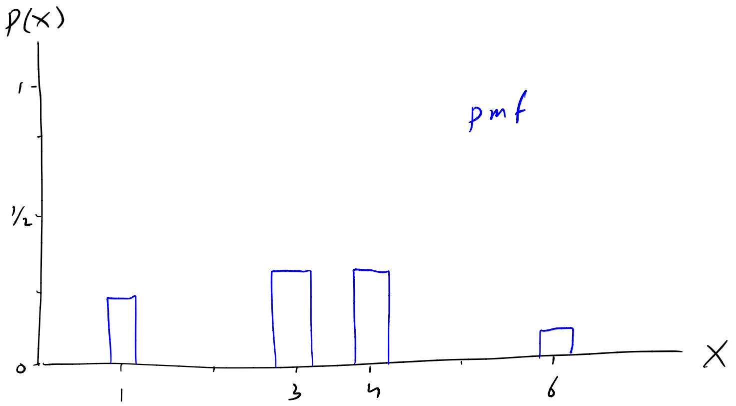
The random variable X has the following probability distribution.

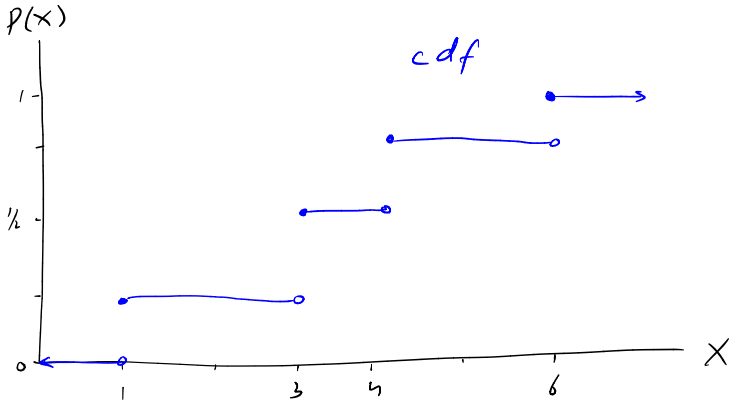
x_i	1	3	4	6
p_i	0.25	0.3	0.3	0.15

The CDF of X is:

$$F(x) = \begin{cases} 0 & \text{if } x < 1 \\ 0.25 & \text{if } 1 \leq x < 3 \\ 0.55 & \text{if } 3 \leq x < 4 \\ 0.85 & \text{if } 4 \leq x < 6 \\ 1 & \text{if } 6 \leq x \end{cases}$$

Plots of the PMF and CDF of X look like this:





Calculating the probability $P(2 \leq X \leq 5)$ from the CDF goes like this:

$$P(2 \leq X \leq 5) \;=\; P((X = 3) \cup (X = 4)) \;=\; F(5) - F(2) \;=\; 0.85 - 0.25 \;=\; 0.6.$$

Task C

The guessing of correct answers by a single student can be seen as a binomial experiment composed of $n = 4$ trials (4 questions). Since a student is randomly guessing each answer, the 4 trials are independent and the probability of success (i.e. guessing the correct answer) in each trial is $1/4$.

Let random variable X be the number of correct answers guessed by a single student. This quantity is distributed according to a binomial distribution $X \sim \mathrm{Bin}(n = 4, p = 1/4)$. Therefore the probability that $X = x$ for $x = 0, 1, 2, 3, 4$ is

$$P(X = x) \;=\; \binom{4}{x}(1/4)^x(3/4)^{4-x}\,.$$

Let random variable Y_x be the number of exam scripts with exactly x correct answers. Then these are also binomially distributed $Y_x \sim \mathrm{Bin}(n = 256, p = P(X = x))$ for $x = 0, 1, \ldots, 4$. We deduce that their expected value $\mathrm{E}(Y_x) = 256 \cdot P(X = x)$ and they are distributed over the values of x as follows:

$$256 \cdot P(X = x) = \begin{cases} 81 & x = 0 \\ 108 & x = 1 \\ 54 & x = 2 \\ 12 & x = 3 \\ 1 & x = 4 \end{cases}$$

Consider the scripts where every answer is correct. Among 256 students we "expect" one of these. But the actual number is a binomially distributed random variable, $Y_4 \sim \mathrm{Bin}(256, 1/256)$.

$$\begin{aligned} P(Y_4 = 1) &= \binom{256}{1} \cdot \left(\frac{1}{256}\right)^1 \cdot \left(\frac{255}{256}\right)^{255} &&= 0.37 \quad (2\,\text{s.f.}) \\ P(Y_4 < 1) &= P(Y_4 = 0) = \binom{256}{0} \cdot \left(\frac{255}{256}\right)^{256} &&= 0.37 \quad (2\,\text{s.f.}) \\ P(Y_4 > 1) &= 1 - P(Y_4 = 1) - P(Y_4 = 0) &&= 0.26 \quad (2\,\text{s.f.}) \end{aligned}$$

So the chance that no student gets all questions right is 37%, that two or more get all questions right is 26%, and the chance of the "expected" number of exactly one all-correct student script is 37%. It is more likely than not random variable Y_4 will fail to take its expected value. The meaning of "expected" in work on probability does not always align with informal understanding of the word, which can seem surprising.

Task D

Define events A_k and B_k for $k = 1, 2, \ldots$.

$$\begin{aligned} A_k &= \{\text{on the } k\text{th throw Alice gets a 5 or 6}\} \\ B_k &= \{\text{on the } k\text{th throw Bob gets a 5 or 6}\} \end{aligned}$$

The probability that on any throw a player gets a 5 or 6 is $p = \frac{2}{6} = \frac{1}{3}$. Therefore $P(A_k) = P(B_k) = p = \frac{1}{3}$, and $P(A_k^c) = P(B_k^c) = q = 1 - p = \frac{2}{3}$. Alice wins in the event

$$A \;=\; A_1 \cup (A_1^c \cap B_1^c \cap A_2) \cup (A_1^c \cap B_1^c \cap A_2^c \cap B_2^c \cap A_3) \cup \cdots$$

Since the events A_k and B_k are independent and the events A_1 and $A_1^c \cap B_1^c \cap A_2$ and $A_1^c \cap B_1^c \cap A_2^c \cap B_2^c \cap A_3$ are pairwise mutually exclusive:

$$\begin{aligned} P(A) &= p + qqp + qqqqp + \cdots \;=\; p(1 + q^2 + q^4 + \cdots) \\ &= \frac{p}{1-q^2} = 0.6. \end{aligned}$$

Bob wins in the event

$$B \;=\; (A_1^c \cap B_1) \cup (A_1^c \cap B_1^c \cap A_2^c \cap B_2) \cup \cdots$$

Similarly

$$\begin{aligned} P(B) &= qp + qqqp + qqqqqp + \cdots = pq(1 + q^2 + q^4 + \cdots) \\ &= \frac{pq}{1-q^2} = 0.4. \end{aligned}$$

We can also approach this using recursion: note that if Alice fails to throw a 5 or 6 on their first go, and Bob does the same, then we are back to the starting situation. Alice wins either if she succeeds on her first throw or if both Alice and Bob fail and then Alice goes on to win. Those are two mutually exclusive possibilities, which gives us the following equation.

$$P(A) \;=\; \frac{1}{3} + \frac{2}{3} \cdot \frac{2}{3} \cdot P(A)$$

We can solve this algebraically.

$$\begin{aligned} P(A) &= \frac{1}{3} + \frac{4}{9} \cdot P(A) \\ 9P(A) &= 3 + 4P(A) \\ 5P(A) &= 3 \\ P(A) &= 0.6 \end{aligned}$$

A similar equation gives Bob's probability of winning.

$$P(B) \;=\; \frac{2}{3} \cdot \frac{1}{3} + \frac{2}{3} \cdot \frac{2}{3} \cdot P(B)$$

From this $P(B) = 0.4$ again follows algebraically.