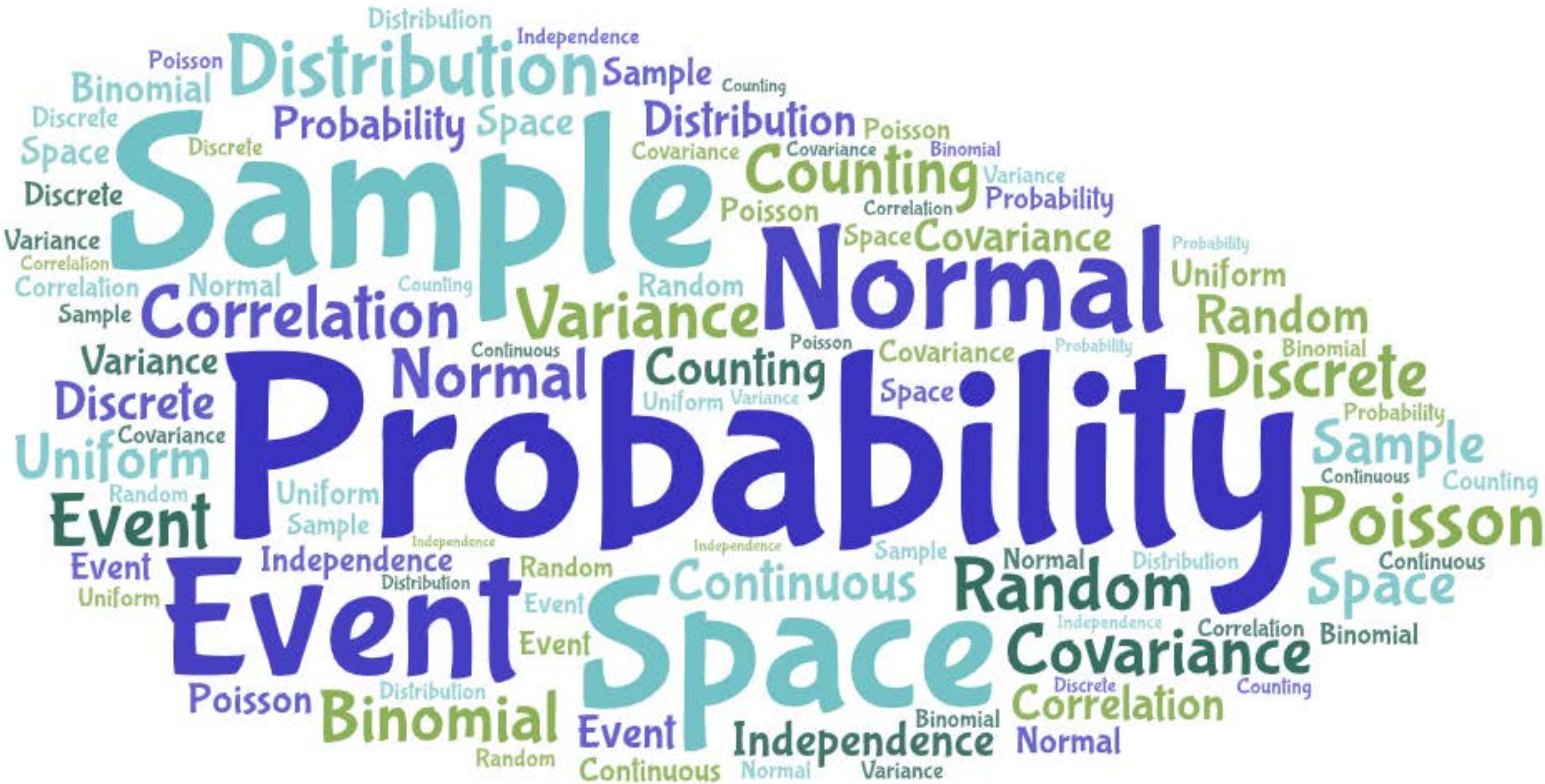
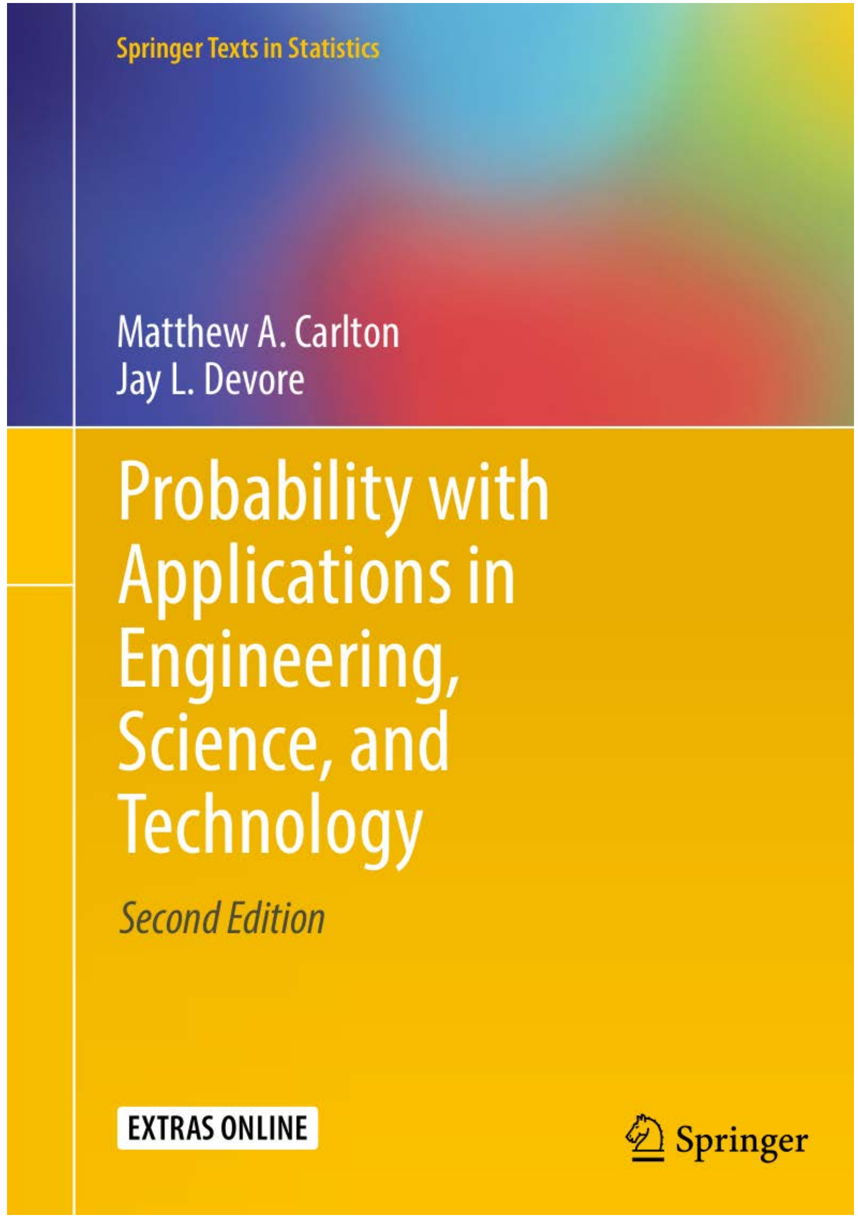
Discrete Mathematics and Probability

Week 7



MARKULF ~~Chris Heunen~~ KOHLWEISS



wooclap.com
code: CMBUZS

E E'

E^c

$\bar{E}$

Topics

- Counting: thinking algorithmically
- Events: what could happen in principle
- Experiments: how can events interact
- Probability: quantifying what could happen

Counting

Counting

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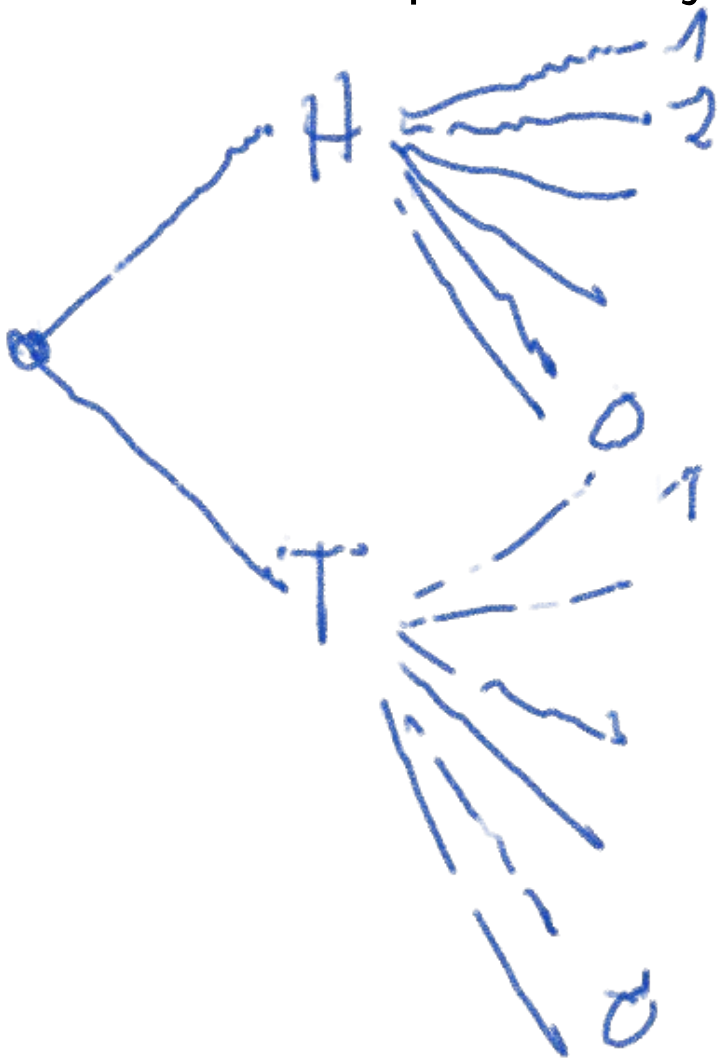
Basic principles of combinatorics:

- if an experiment has n outcomes;
  and another experiment has m outcomes,
- then the two experiments jointly have $n \cdot m$ outcomes.



Permutations

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Definition

Let $H = \{h_1, h_2, \ldots, h_n\}$ be a set of n different objects. The *permutations* of H are the different orders in which you can write all of its elements.

Example: Horse race $\{A, B, C, D, E\}$

$$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5! = 120$$

$$n! = n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1 = \prod_{i=1}^{n} i$$

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e.g. (E, C, B, A, D)

gold silver bronze

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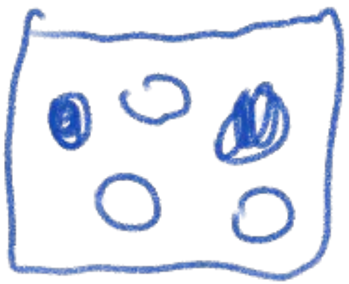
gold silver bronze

How many in general? $n! = n \cdot (n-1) \cdot (n-2) \cdots 1$

Permutations with repetitions

Definition

Let $H = \{h_1 \ldots h_1,\ h_2 \ldots h_2,\ \ldots,\ h_r \ldots h_r\}$ be a set of r different types of repeated objects: n_1 many of h_1, n_2 of h_2, $\ldots n_r$ of h_r.
The *permutations with repetitions* of H are the different orders in which you can write all of its elements.



$$\frac{5!}{3!\,2!}$$

Permutations with repetitions

class question

Definition

Let $H = \{h_1 \dots h_1,\ h_2 \dots h_2,\ \dots,\ h_r \dots h_r\}$ be a set of r different types of repeated objects: n_1 many of h_1, n_2 of h_2, $\dots n_r$ of h_r.
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not all elements are different

have a little think! Maybe more than a little think

$$n = \sum_{i=1}^{r} n_i$$

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$$\frac{n!}{n_1! \cdots n_r!} \qquad n = n_1 + n_2 \cdots n_r$$

k-Permutations

Definition

Let $H = \{h_1, h_2, \ldots, h_n\}$ be a set of n different objects. The *k-permutations* of H are the different ways in which one can pick and write k of its elements of H in order.

How many?

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How many?

Example: top 3 in horse race $\{A, B, C, D, E\}$

(E, C, D) A B

B, A

$$\frac{5!}{2!}$$

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e.g. (E, B, A)

$$\frac{5!}{2!} = 60$$

k-Permutations

Definition

Let $H = \{h_1, h_2, \ldots, h_n\}$ be a set of n different objects. The k-*permutations* of H are the different ways in which one can pick and write k of its elements of H in order.

How many? $P_{n,k} = \dfrac{n!}{(n-k)!}$ $\quad nHk$

Example: top 3 in horse race $\{A, B, C, D, E\}$

e.g. (E, B, A)

$$\frac{5!}{2!} = 60$$

k-Permutations with repetitions

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Definition

Let $H = \{h_1 \ldots, h_2 \ldots, \ldots, h_r \ldots\}$ be a set of r different types of repeated objects, each of infinite supply. The *k-permutations with repetitions* of H are the different orders in which one can write an ordered sequence of length k using the elements of H.

Again set with repeated objects, but now each of infinite supply.

$26 \cdot 26 \cdot 26 = 26^3$ $\qquad r^k$

k-Permutations with repetitions

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Example: # 3-letter words of letters English Alphabet $= 26^3 = 26 \cdot 26 \cdot 26$
17576

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k-bit strings $= 2^k$

k-Combinations

Definition

Let $H = \{h_1, h_2, \ldots, h_n\}$ be a set of n different objects. The *k-combinations* of H are the different ways in which one can pick k of its elements without order.

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How many?

$\underbrace{A, B, C, D, E, F}$, ...

$$\frac{50!}{6!\,44!}$$

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How many?

$$C_{n,k} = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

"binomial coefficient"

$$C_{n,k} = \frac{P_{n,k}}{k!}$$

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Example: # ways to form committee of 5 students from class of 30

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"binomial coefficient" $C_{n,k} = \frac{P_{n,k}}{k!}$

Example: # ways to form committee of 5 students from class of 30

$= \binom{30}{5} = \frac{30!}{5!25!} = 142506$

Now you know how to count.

Events

Events

A mathematical model for experiments:

- *Sample space*: the set Ω of all possible outcomes
- An *event* is a collection[1] of possible outcomes: $E \subseteq \Omega$
- *Union* $E \cup F$ and *intersection* $E \cap F$ of events make sense

[1] Sometimes Ω is too large, and not all subsets are events. Ignore this now.

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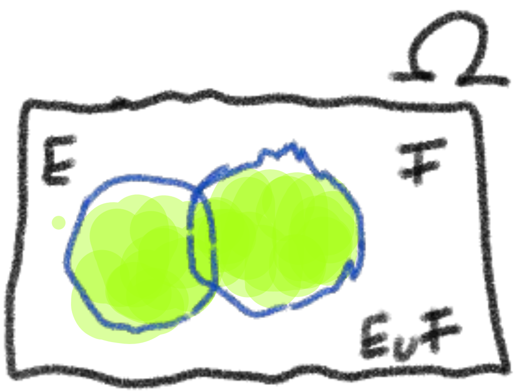

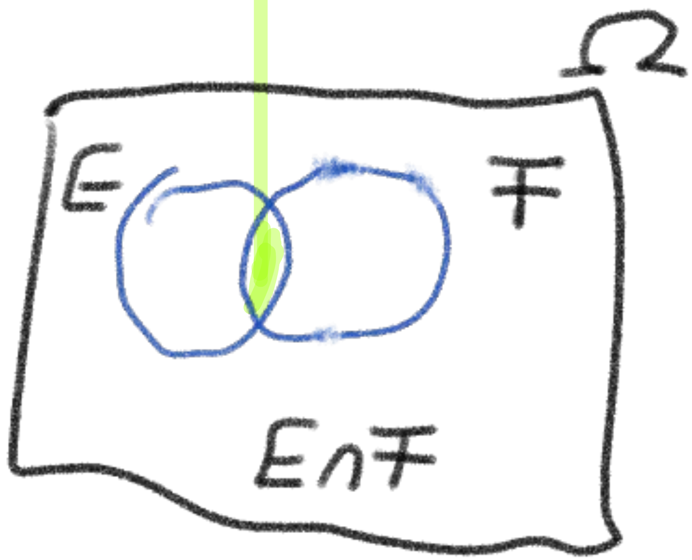

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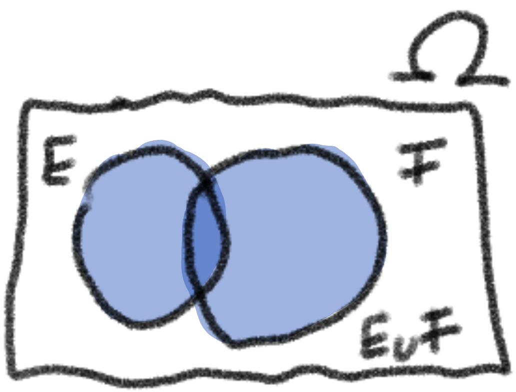

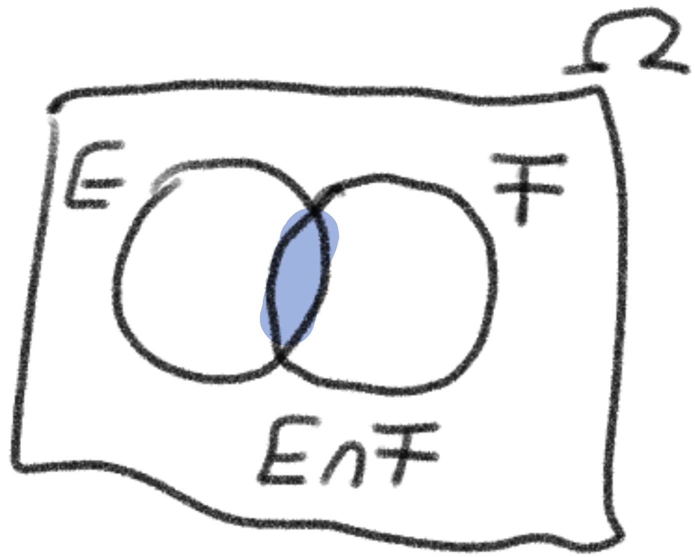

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Examples

Experiment	Event Space	Event
Will it rain	{r, n}	{r}
Throw a dice	{1, 2 ... 6}	{2, 4, 6}

Examples

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Will it rain?	$\Omega = \{r, n\}$	e.g. $E = \{r\}$

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5-horse race	$\Omega =$ permutations of $\{A, B, C, D, E\}$	e.g. $E = \{B \text{ wins}\}$ $F = \{E \text{ wins}, A \text{ third}\}$
flip 2 coins	$\Omega = \{(H,H), (H,T), (T,H), (H,H)\}$	e.g. $E =$ coins different $= \{(H,T)\ (T,H)\}$

Examples

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Will it rain?	$\Omega = \{r, n\}$	e.g. $E = \{r\}$
5-horse race	Ω = permutations of $\{A, B, C, D, E\}$	e.g. $E = \{B \text{ wins}\}$ $F = \{C \text{ wins}, A \text{ third}\}$
flip 2 coins	$\Omega = \{(H,H), (H,T), (T,H), (H,H)\}$	e.g. E = coins different $= \{(H,T), (T,H)\}$ F = at least one head $= \{(H,T), (T,H), (H,H)\}$
roll dice until 1 get 6	$\Omega = \{$sequences of 1–5 and then $6\}$	E = first 4, then 6 on 3rd roll $= \{(4,1,6), (4,2,6), (4,3,6) \dots\}$

Union and intersection

Union

Union $E \cup F$ of events E and F means E ~~and~~ or F.
Infinite union $\bigcup_i E_i$ of events E_i means at least one of the E_i's.

Intersection

Intersection $E \cap F$ of events E and F means E and F.
Infinite intersection $\bigcap_i E_i$ of events E_i means each of the E_i's.

Definition

If $E \cap F = \emptyset$, we call events E and F *mutually exclusive*.
If events $E_1, E_2, \ldots$ satisfy $E_i \cap E_j = \emptyset$ whenever $i \neq j$, we call them *mutually exclusive*. They cannot happen at the same time.

Inclusion and implication

Remark

If the event E is a *subset* of the event F, written $E \subseteq F$, then the occurrence of E *implies* that of F.

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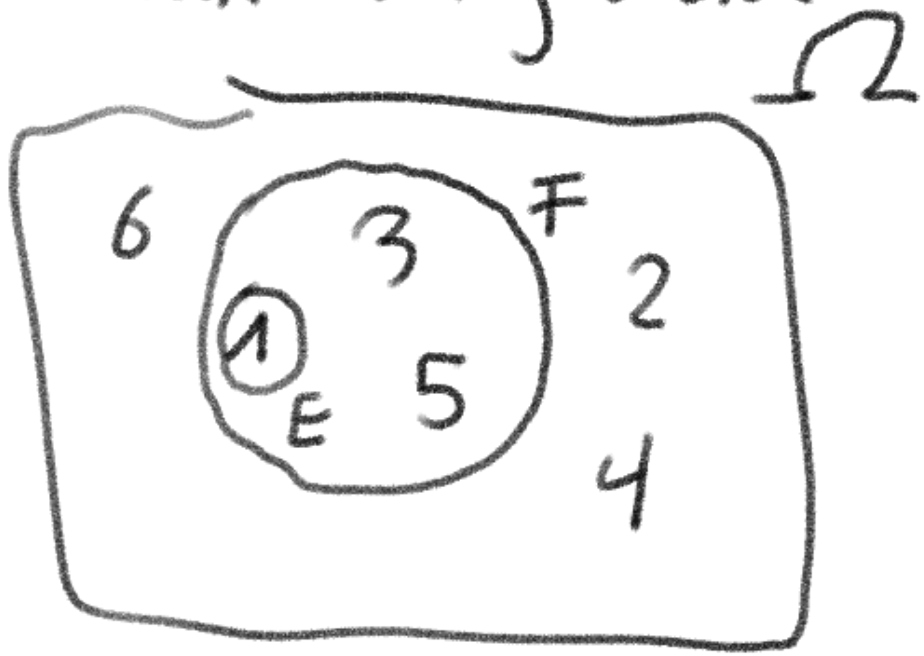
Experiment: rolling a dice



$E = \{\text{rolling } 1\} \subseteq \{\text{roll odd number}\} = F$

Complementarity

Definition

The *complement* of an event E is $E^c = \Omega - E$.
This is the event that E does *not* occur.

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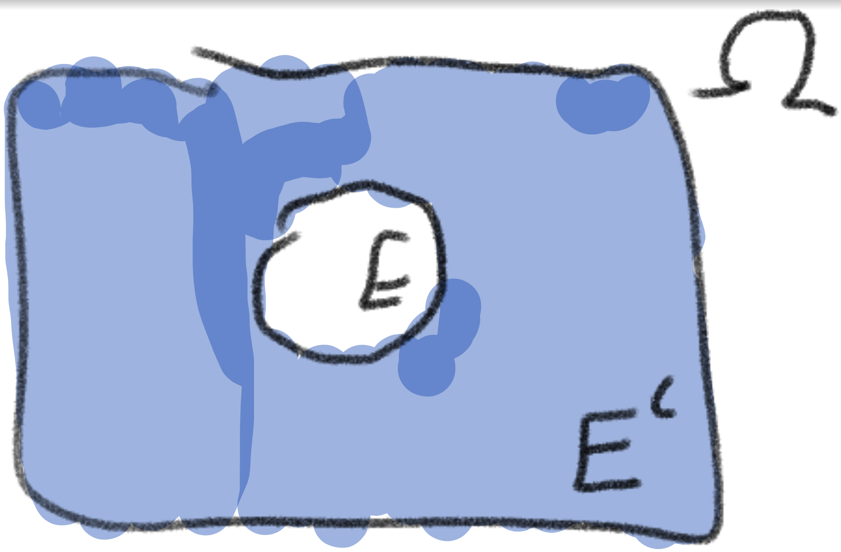

$E \cap E^c = \emptyset \qquad E \cup E^c = \Omega$

Experiments

Experiments

How events can interact:

- Commutativity
- Distributivity
- Associativity
- De Morgan's Law

Properties of events

- Commutativity: $E \cup F = F \cup E$
  $E \cap F = F \cap E$

- Associativity: $E \cup (F \cup G) = (E \cup F) \cup G$
  $E \cap (F \cap G) = (E \cap F) \cap G$

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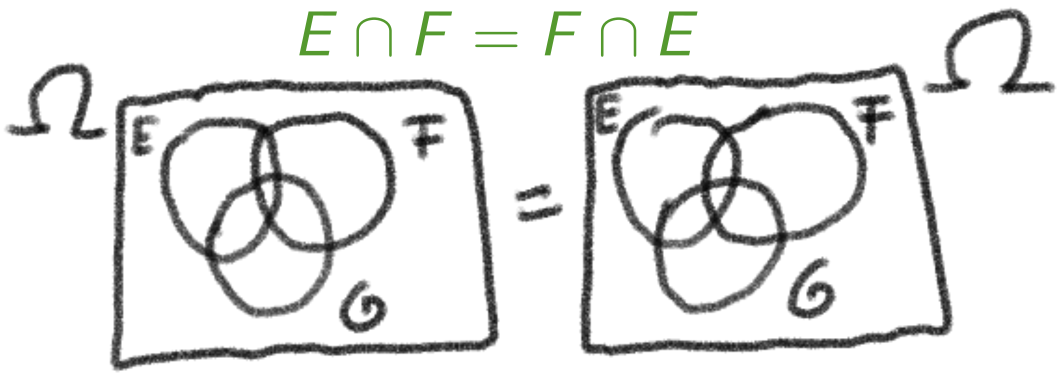

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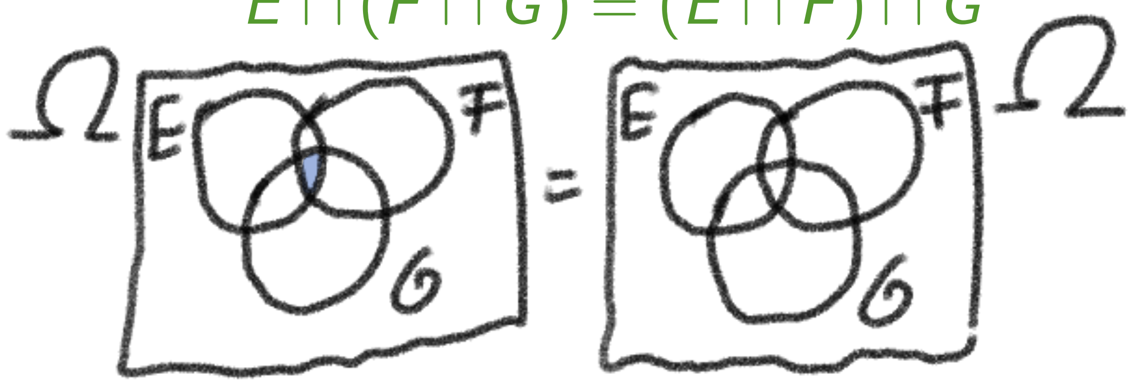

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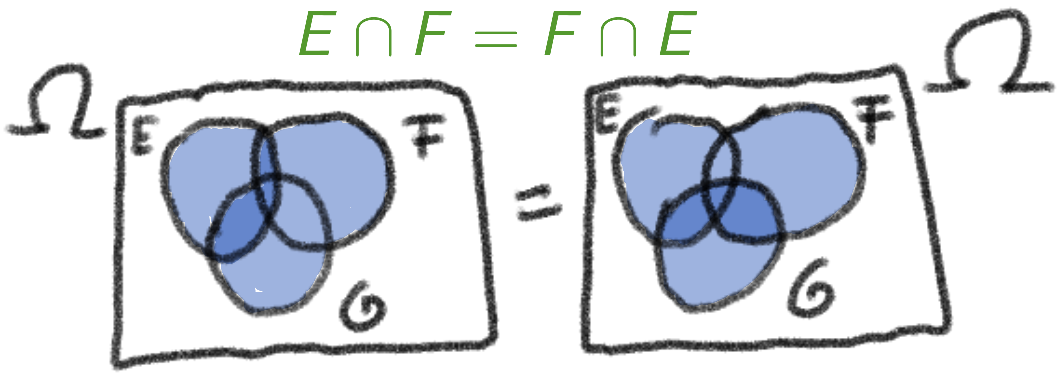

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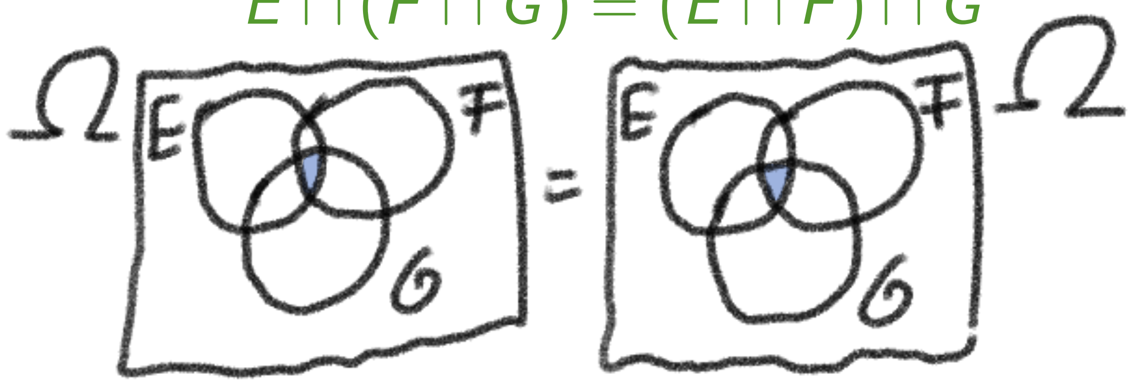

Properties of events

- Distributivity: $(E \cup F) \cap G = (E \cap G) \cup (F \cap G)$
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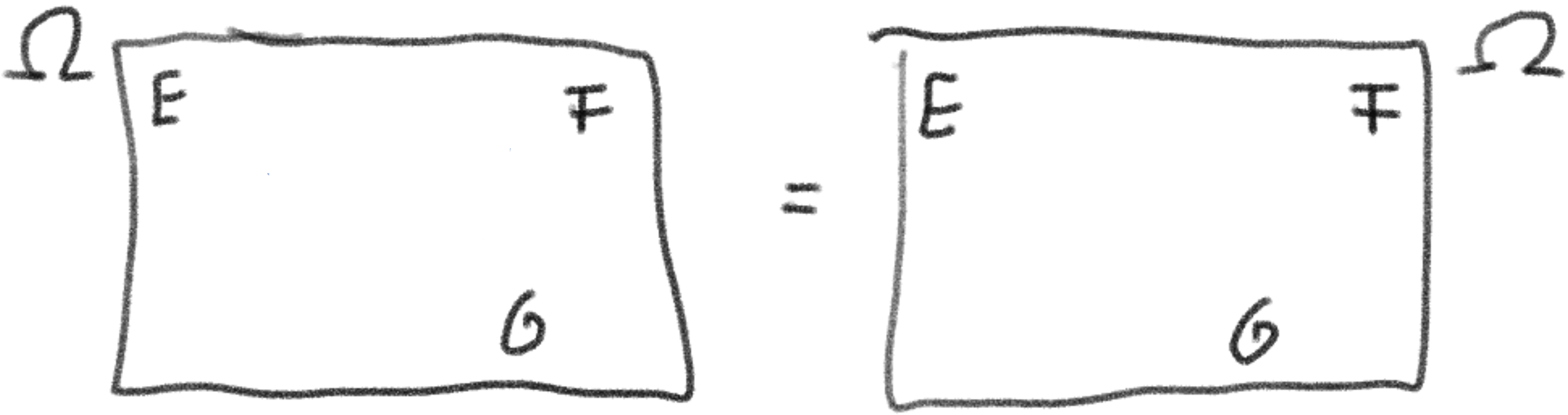

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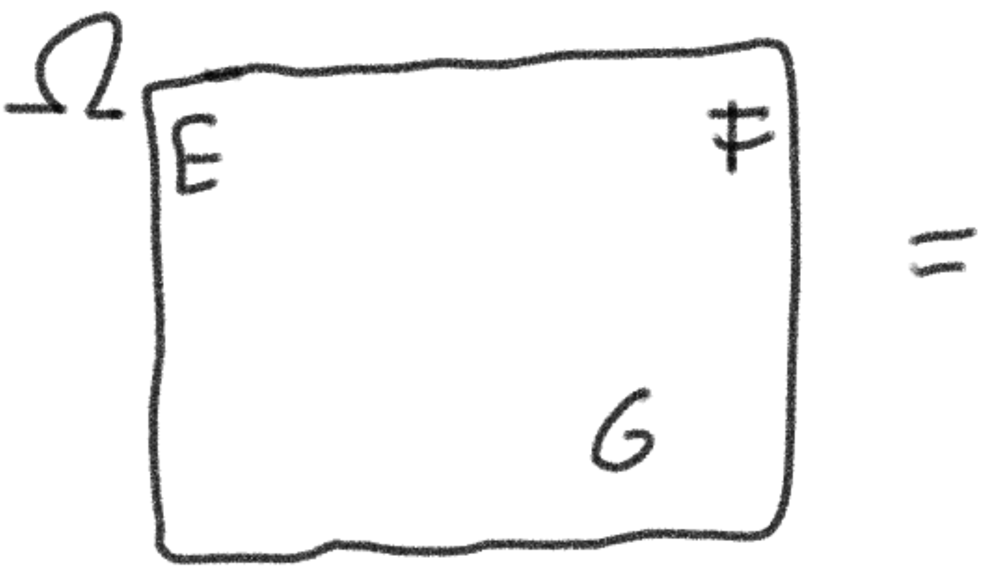

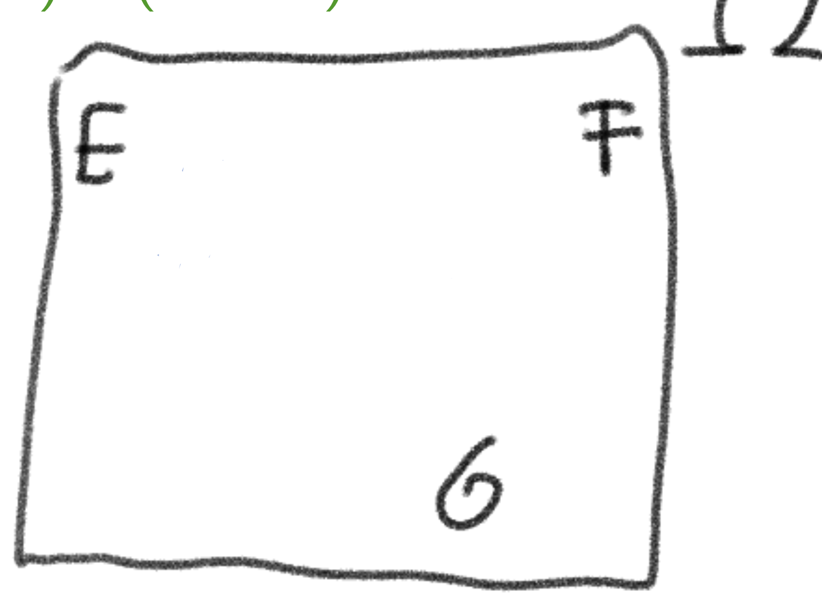

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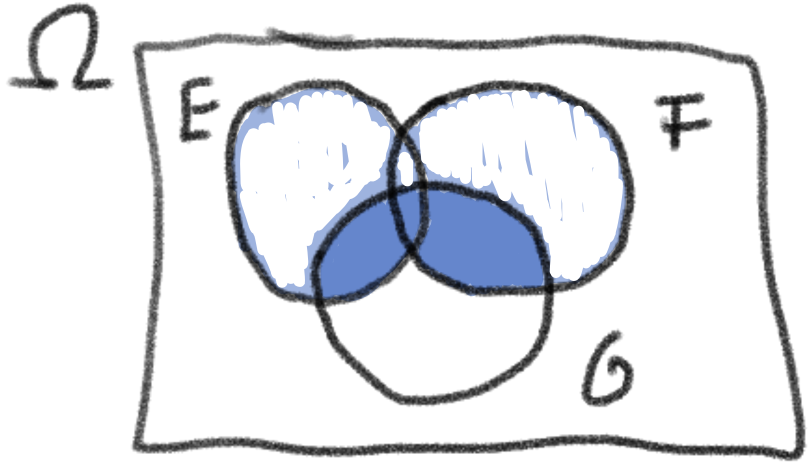

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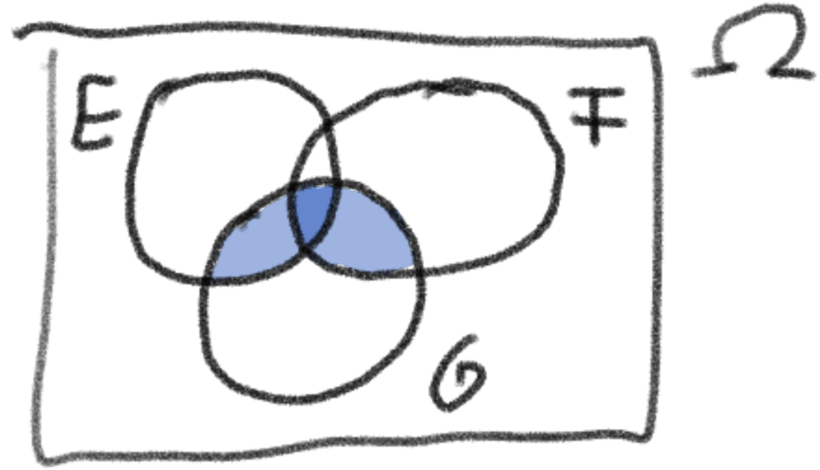

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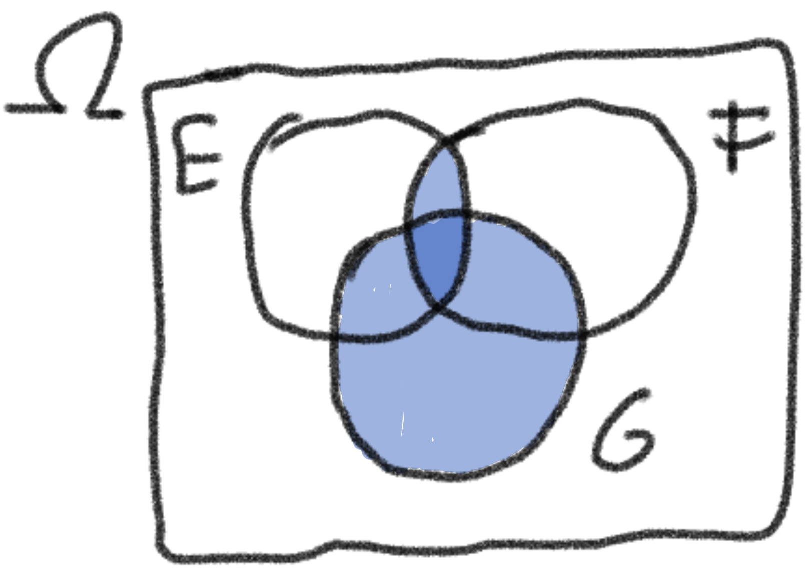

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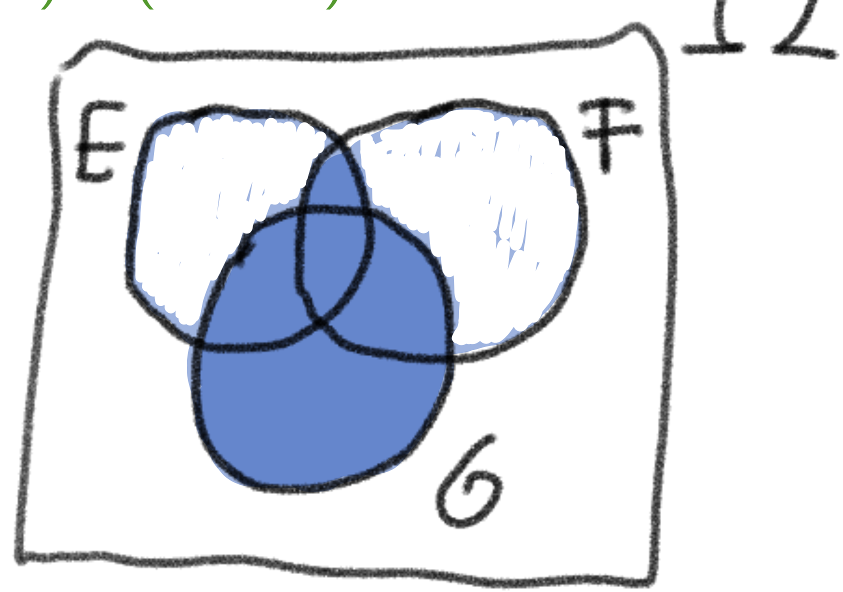

De Morgan's law

- De Morgan's law: $(E \cup F)^c = E^c \cap F^c$

De Morgan's law

E= you have umbrella
F= I have umbrella

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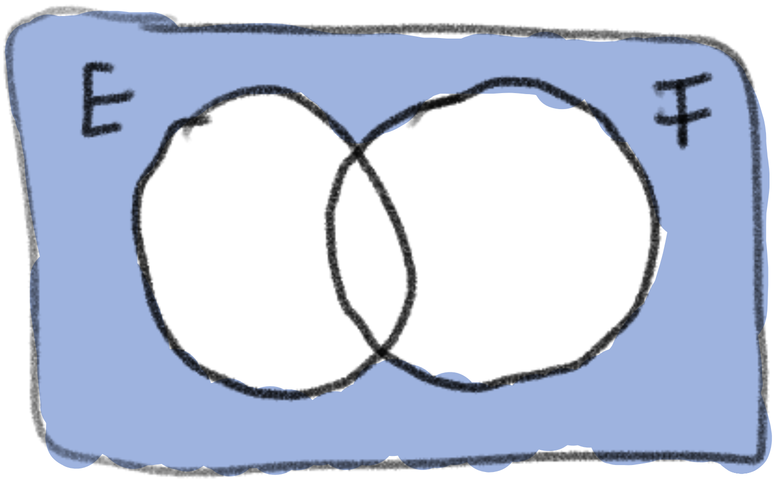

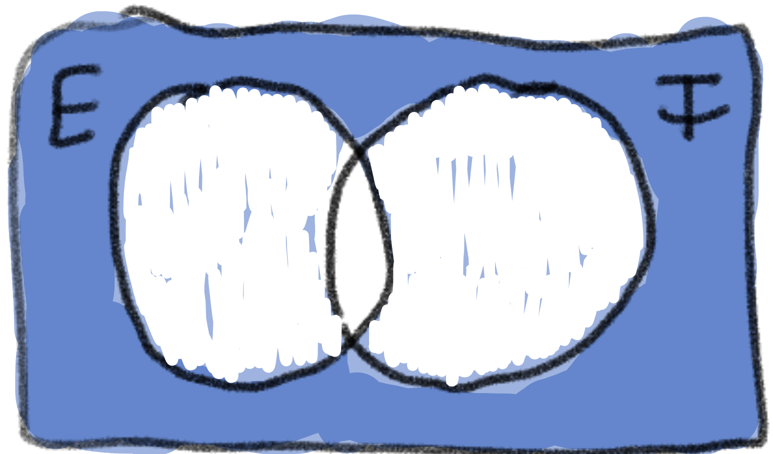

$(E \cap F)^c = E^c \cup F^c$ = not true both have umbrella = one does not have umbrella

De Morgan's law

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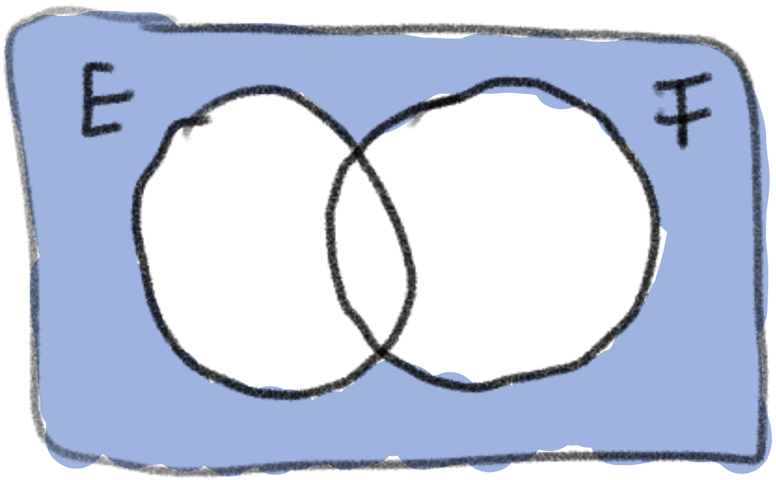

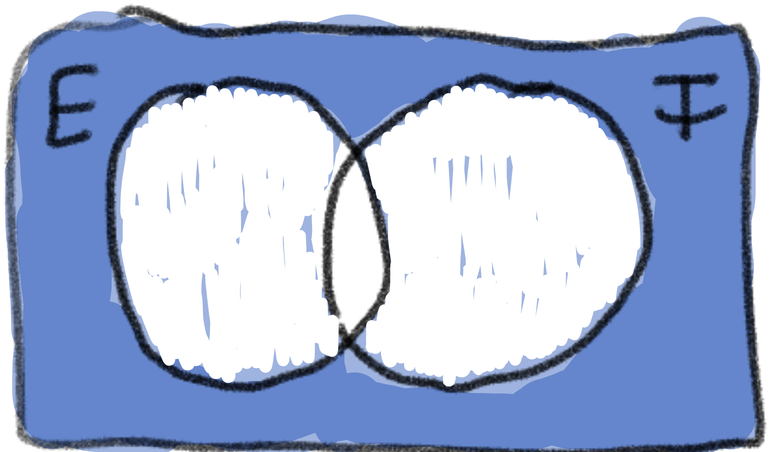

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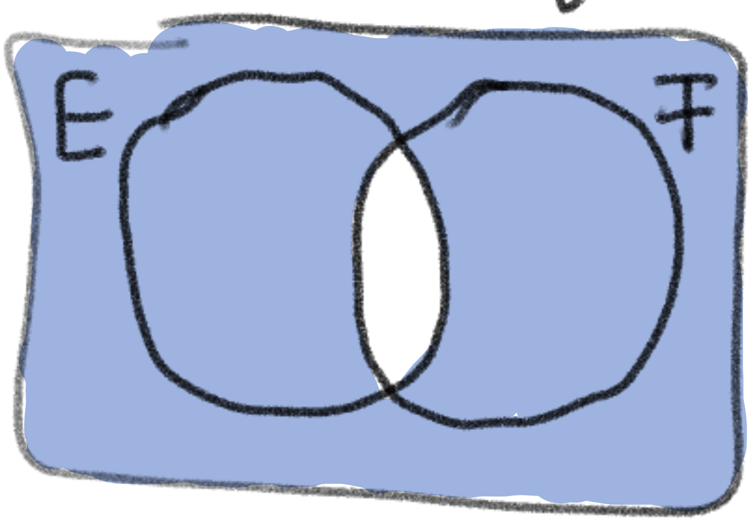

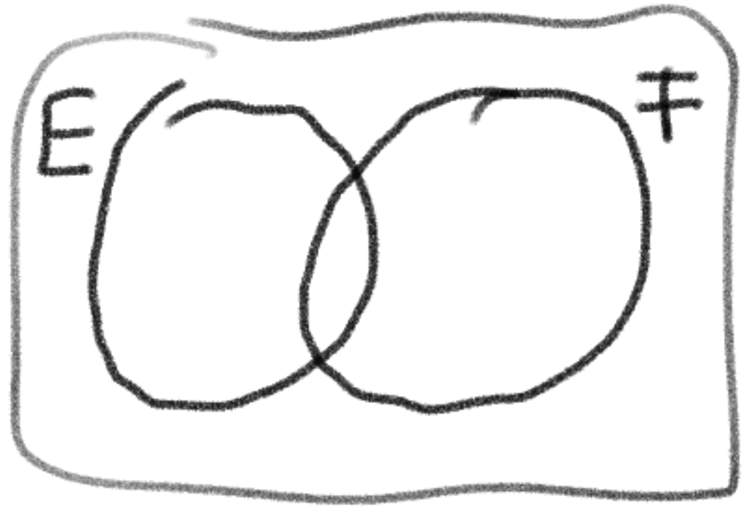

Probability

So far been structural
now we will attach numbers to things

- Definition by axioms
- How to compute probabilities
- Inclusion-exclusion principle
- Equally likely outcomes

Axioms of probability

Definition

The *probability* $\mathbf{P}$ on a sample space Ω assigns *numbers* to *events* of Ω in such a way that:

1. the probability of any event is non-negative: $\mathbf{P}(E) \geq 0$;
2. the probability of the sample space is one: $\mathbf{P}(\Omega) = 1$;
3. for countably many *mutually exclusive* events $E_1, E_2, \ldots$:

$$\mathbf{P}\Big(\bigcup_i E_i\Big) = \sum_i \mathbf{P}(E_i)$$

Axioms of probability

Definition P is a function that satisfies 3 axioms

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$$\mathbf{P}\Big(\bigcup_i E_i\Big) = \sum_i \mathbf{P}(E_i)$$

finite $P(E_1 \cup \cdots \cup E_n) = P(E_1) + \cdots + P(E_n)$

countably infinite $P(E_1 \cup E_2 \cup E_3 \cup \ldots) = P(E_1) + P(E_2) + P(E_3) + \cdots$

How to compute probabilities

Proposition $P(E) = 1 - P(E^c)$

For any event, $\mathbf{P}(E^c) = 1 - \mathbf{P}(E)$.

1) $E \cap E^c = \emptyset$ thus by axiom 3 $P(E) + P(E^c) = P(E \cup E^c) = P(\Omega) = 1$

2) $P(\Omega) = 1$ thus by axiom 2

$$P(\emptyset) = P(\Omega^c) = 1 - P(\Omega) = 1 - 1 = 0$$

Corollary

We have $\mathbf{P}(\emptyset) = \mathbf{P}(\Omega^c) = 1 - \mathbf{P}(\Omega) = 1 - 1 = 0$.
For any event, $\mathbf{P}(E) = 1 - \underbrace{\mathbf{P}(E^c)}_{>0} \leq 1$.

How to compute probabilities

Proposition

For any event, $\mathbf{P}(E^c) = 1 - \mathbf{P}(E)$.

Proof: E and E^c are mutually exclusive $E \cap E^c = \emptyset$

so by $\underline{\text{axiom 3}}$ $P(E) + P(E^c) = P(\Omega)$

by $\underline{\text{axiom 2}}$ $= 1$

Corollary

1) *We have* $\mathbf{P}(\emptyset) = \mathbf{P}(\Omega^c) = 1 - \mathbf{P}(\Omega) = 1 - 1 = 0$.

2) *For any event,* $\mathbf{P}(E) = 1 - \underbrace{\mathbf{P}(E^c)}_{\text{by } \underline{\text{axiom 1}} \geq 0} \leq 1$.

How to compute probabilities

Proposition

For any two events, $\mathbf{P}(E \cup F) = \mathbf{P}(E) + \mathbf{P}(F) - \mathbf{P}(E \cap F)$.

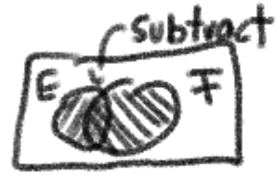
Proposition (*Boole's inequality*)

For any events $E_1, E_2, \ldots, E_n$:

$$\mathbf{P}\left(\bigcup_{i=1}^{n} E_i\right) \leq \sum_{i=1}^{n} \mathbf{P}(E_i).$$

skipping proof by induction

How to compute probabilities



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skipping proof by induction initially

Inclusion-exclusion

Proposition

For any events:

$$\begin{aligned}\mathbf{P}(E \cup F \cup G) = {} & \mathbf{P}(E) + \mathbf{P}(F) + \mathbf{P}(G) \\ & - \mathbf{P}(E \cap F) - \mathbf{P}(E \cap G) - \mathbf{P}(F \cap G) \\ & + \mathbf{P}(E \cap F \cap G).\end{aligned}$$

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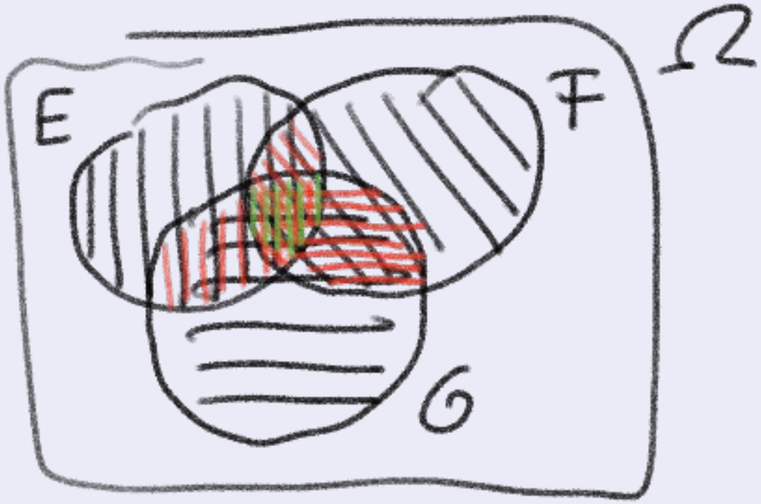

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$$\begin{aligned}\mathbf{P}(E_1 \cup E_2 \cup \cdots \cup E_n) = {} & \sum_{1 \le i \le n} \mathbf{P}(E_i) \\ & - \sum_{1 \le i_1 < i_2 \le n} \mathbf{P}(E_{i_1} \cap E_{i_2}) \\ & + \sum_{1 \le i_1 < i_2 < i_3 \le n} \mathbf{P}(E_{i_1} \cap E_{i_2} \cap E_{i_3}) \\ & - \cdots \\ & + (-1)^{n+1} \mathbf{P}(E_1 \cap E_2 \cap \cdots \cap E_n).\end{aligned}$$

all the way up to n

Example

wooclap

Example

In a sports club, with N members

36 members play tennis, 22 play tennis and squash,
28 play squash, 12 play tennis and badminton,
18 play badminton, 9 play squash and badminton,
4 play tennis, squash and badminton.

What is probability that a random member
~~How many~~ plays at least one of these games?

Example

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$$P(T \cup S \cup B) = P(T) + P(S) + P(B) - P(T \cap S) - P(T \cap B) - P(S \cap B) + P(T \cap S \cap B)$$

$$= \frac{36}{N} + \frac{28}{N} + \frac{18}{N} - \frac{22}{N} - \frac{12}{N} - \frac{9}{N} + \frac{4}{N} = \frac{43}{N}$$

How to compute probabilities

Proposition

If $E \subseteq F$, *then* $\mathbf{P}(F - E) = \mathbf{P}(F) - \mathbf{P}(E)$.

Corollary

If $E \subseteq F$, *then* $\mathbf{P}(E) \leq \mathbf{P}(F)$.

Equally likely outcomes

The return of counting

The last thing before we all go home
I want to talk about this

Finite sample space, $|\Omega| = N < \infty$, has special important case where each experiment outcome has *equal probability*:

$$\mathbf{P}(\omega) = \frac{1}{N} \qquad \text{for all } \omega \in \Omega$$

Definition

Outcomes $\omega \in \Omega$ are also called *elementary events*.

Example

Example

Rolling two dice, what is the probability that the sum of the numbers shown is 7?

What's wrong with this solution? "The number 7 is one out of the possible values $2, 3, \ldots, 12$ for the sum, and the answer is $\frac{1}{11}$."

what is wrong with the answer?

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$$\Omega = \{(i,j) \mid i,j \in \{1,2,\dots,6\}\}$$

$$E = \text{sum is } 7 = \{(i,j) \mid i+j=7\} = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$$

$$P(E) = \frac{6}{36} = \frac{1}{6}$$

Summary

- Counting: permutations, combinations, repetitions
- Events: sample space, union, intersection, complement
- Experiments: distributivity, De Morgan's law
- Probability: axioms, how to compute, equally likely outcomes