Discrete Mathematics and Probability — **Induction and Recursion**

Session 2025/26, Semester 1 — Week 3 Homework 2

This homework runs from Thursday 25 September 2025 until 12 noon on Thursday 2 October 2025. Submission is to Gradescope Homework 2.

Questions marked with an asterisk $*$ may be a little harder than others. All are still within the course curriculum, though, and can be done using the methods taught in the study guides and textbook.

You should aim to write out solutions that someone who does not already know the answer could follow and understand.

You will receive marks and feedback on this homework, but this is formative assessment and does not affect your final course grade. You may discuss your work with others and can ask questions about the homework in lectures, to your tutors, and in Piazza: please do.

Question 1

(a) For which positive integers n is $3^n < n!$?

(b) Prove by induction your statement in (a).

[*3 marks*]

Question 2

Suppose a sequence of integers $a_0, a_1, a_2, \ldots$ is defined recursively as follows:

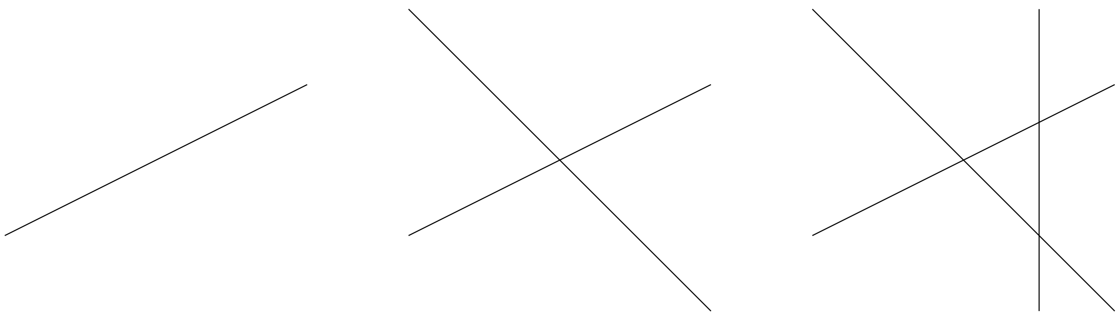
$$a_0 = 0 \quad \text{and} \quad a_n = 3a_{n-1} + 4 \quad \text{for } n \geq 1.$$

Prove by induction that $a_n = 2(3^n - 1)$ for all integers $n \geq 0$.

[*3 marks*]

$*$ Question 3

The diagrams below show a plane divided into regions by different numbers of straight lines. Suppose r_n is the maximum number of regions a plane can be divided into by n straight lines.



$r_1 = 2$ $\qquad$ $r_2 = 4$ $\qquad$ $r_3 = 7$

(a) Write down a recurrence relation expressing r_{n+1} in terms of r_n for $n \geq 1$, with an explanation justifying it.

(b) Use this to prove by induction that $r_n = \dfrac{n^2 + n + 2}{2}$.

[*4 marks*]