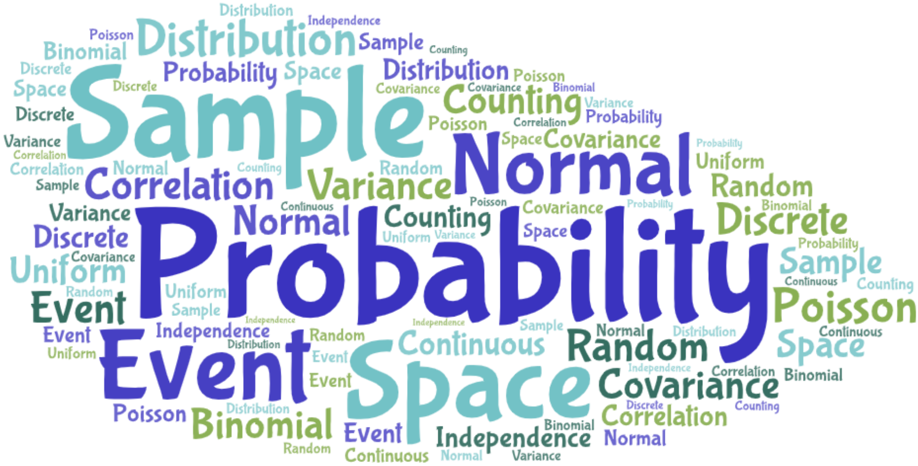
Discrete Mathematics and Probability

Week 8



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Topics

- Independence: what information changes probability
- Random variables: when variables depend on chance
- Expectation: most likely outcomes of experiment
- Variance: how much the experiment can deviate

Independence

Independence

Sometimes partial information on an experiment
does not change the likelihood of an event.

Definition

Events E and F are *independent* if $\mathbf{P}(E \mid F) = \mathbf{P}(E)$.
Equivalently: $\mathbf{P}(E \cap F) = \mathbf{P}(E) \cdot \mathbf{P}(F)$.
Equivalently: $\mathbf{P}(F \mid E) = \mathbf{P}(F)$.

Proposition

*If E and F are independent events,
then E and F^c are also independent.*

Examples

Independence

Definition

Three events E, F, G are *(mutually) independent* if:

$$\begin{aligned}
\mathbf{P}\{E \cap F\} &= \mathbf{P}\{E\} \cdot \mathbf{P}\{F\}, \\
\mathbf{P}\{E \cap G\} &= \mathbf{P}\{E\} \cdot \mathbf{P}\{G\}, \\
\mathbf{P}\{F \cap G\} &= \mathbf{P}\{F\} \cdot \mathbf{P}\{G\}, \\
\mathbf{P}\{E \cap F \cap G\} &= \mathbf{P}\{E\} \cdot \mathbf{P}\{F\} \cdot \mathbf{P}\{G\}.
\end{aligned}$$

For more events the definition is that any (finite) subset of them have this factorisation property.

Examples

Random variables

Random variables

"Random variables $\approx$ random numbers". But *random* means that there must be some kind of experiment behind these numbers.

Definition

A *random variable* is a function from the sample space Ω to the real numbers $\mathbb{R}$.

Discrete random variables

Definition

A random variable X that can take on countably many possible values is called *discrete*.

Probability mass function

Definition

The *probability mass function (pmf)*, or *distribution* of a discrete random variable X gives the probabilities of its possible values:

$$\mathfrak{p}_X(x_i) = \mathbf{P}(X = x_i),$$

Proposition

$$\mathfrak{p}(x_i) \geq 0 \qquad \textit{and} \qquad \sum_i \mathfrak{p}(x_i) = 1$$

Examples

Cumulative distribution function

Definition

The *cumulative distribution function (cdf)* of a random variable X:

$$F\colon \mathbb{R} \to [0, 1], \qquad x \mapsto F(x) = \mathbf{P}(X \leq x).$$

Cumulative distribution function

Proposition

A cumulative distribution function F:

- *is non-decreasing: if $x \leq y$ then $F(x) \leq F(y)$*
- *has limit $\lim_{x \to -\infty} F(x) = 0$ on the left*
- *has limit $\lim_{x \to \infty} F(x) = 1$ on the right*

Expectation

Expectation

Once we have a random variable, we want to quantify its *typical* behaviour in some sense. Two of the most often used quantities for this are the *expectation* and the *variance*.

Definition

The *expectation* of a discrete random variable X is:

$$\mathbf{E}X = \sum_i x_i \cdot \mathfrak{p}(x_i)$$

provided the sum exists. Also called *mean*, or *expected value*.

Examples

Properties of expectation

Proposition (expectation of a function of a random variable)

If X is a discrete random variable, and $g : \mathbb{R} \to \mathbb{R}$ a function, then:

$$\mathbf{E}g(X) = \sum_i g(x_i) \cdot \mathfrak{p}(x_i) \qquad \textit{(if it exists)}$$

Corollary (expectation is linear)

If X is a discrete random variable, and a, b fixed real numbers:

$$\mathbf{E}(aX + b) = a \cdot \mathbf{E}X + b.$$

Moments

Definition (moments)

Let $n \in \mathbb{N}$. The n^{th} *moment* of a random variable X is:

$$\mathbf{E}X^n$$

The n^{th} *absolute moment* of X is:

$$\mathbf{E}|X|^n$$

Variance

Example

3. Variance

Definition (variance, standard deviation)

The *variance* and the *standard deviation* of a random variable are:

- $\mathbf{Var}\, X = \mathbf{E}(X - \mathbf{E}X)^2$.
- $\mathrm{SD}\, X = \sqrt{\mathbf{Var}\, X}$.

Example

Properties of the variance

Proposition (equivalent form of the variance)

$\mathbf{Var}\, X = \mathbf{E}X^2 - (\mathbf{E}X)^2$ *for any random variable* X.

Corollary

$\mathbf{E}X^2 \geq (\mathbf{E}X)^2$ *for any random variable* X,
with equality only if X *is constant.*

Examples

Properties of the variance

Proposition (variance is not linear)

Let X be a random variable, a and b fixed real numbers. Then:

$$\mathbf{Var}(aX + b) = a^2 \cdot \mathbf{Var}\, X$$

Summary

- Independence: what information changes probability
- Random variables: when variables depend on chance
- Expectation: most likely outcomes of experiment
- Variance: how much the experiment can deviate