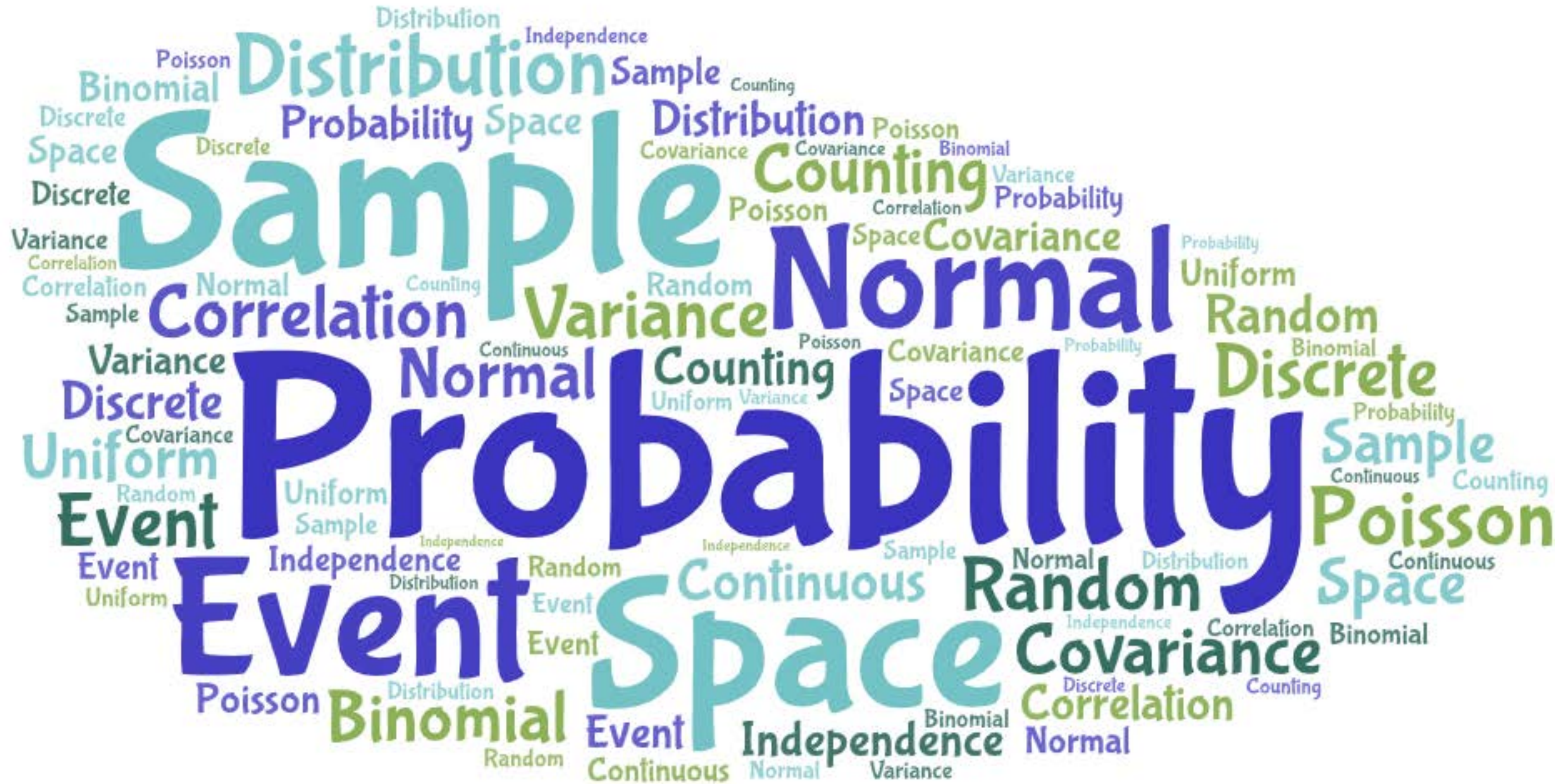
Discrete Mathematics and Probability

Week 8



MARKULF ~~Chris Heunen~~ KOHLWEISS

Probability mass function

Definition

The *probability mass function (pmf)*, or *distribution* of a discrete random variable X gives the probabilities of its possible values:

$$\mathfrak{p}_X(x_i) = \mathbf{P}(X = x_i),$$

Proposition

$$\mathfrak{p}(x_i) \geq 0 \qquad \textit{and} \qquad \sum_i \mathfrak{p}(x_i) = 1$$

Examples

Fix parameter $\lambda > 0$

Define $p(i) = c \cdot \frac{\lambda^i}{i!}$

1) $p(i) > 0$ iff $c > 0$

2) $\sum_i p(i) = 1$

$$\sum_i c \cdot \frac{\lambda^i}{i!} = c \overbrace{\sum_i \frac{\lambda^i}{i!}}^{e^{\lambda}} = \overset{c}{\overset{\|}{e^{-\lambda}}} \cdot e^{\lambda} = 1$$

$$P(X > 1) = 1 - P(X=0) - P(X=1)$$
$$= 1 - e^{-\lambda} - e^{-\lambda} \cdot \lambda$$

Examples

Fix parameter $\lambda > 0$

Define $p(i) = c \cdot \frac{\lambda^i}{i!}$

which c makes this a pmf?

$p(i) \geq 0 \iff c \geq 0$

Examples

Fix paramater $\lambda > 0$

Define $p(i) = c \cdot \frac{\lambda^i}{i!}$

Taylor or Maclaurin Series

which c makes this a pmf. ?

$p(i) \geqslant 0 \iff c \geqslant 0$

$$\sum_{i=0}^{\infty} p(i) = \sum_{i=0}^{\infty} c \cdot \frac{\lambda^i}{i!} = c \cdot \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = c \cdot e^{\lambda} \iff c = e^{-\lambda}$$

Examples

Fix parameter $\lambda > 0$

Define $p(i) = c \cdot \dfrac{\lambda^i}{i!}$

which c makes this a pmf?

Taylor or Maclaurin Series

$$p(i) \geq 0 \iff c \geq 0$$

$$\sum_{i=0}^{\infty} p(i) = \sum_{i=0}^{\infty} c \cdot \frac{\lambda^i}{i!} = c \cdot \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = c \cdot e^{\lambda} \iff c = e^{-\lambda}$$

so e.g. $P(X=0) = p(0) = e^{-\lambda} \cdot \dfrac{\lambda^0}{0!} = e^{-\lambda}$

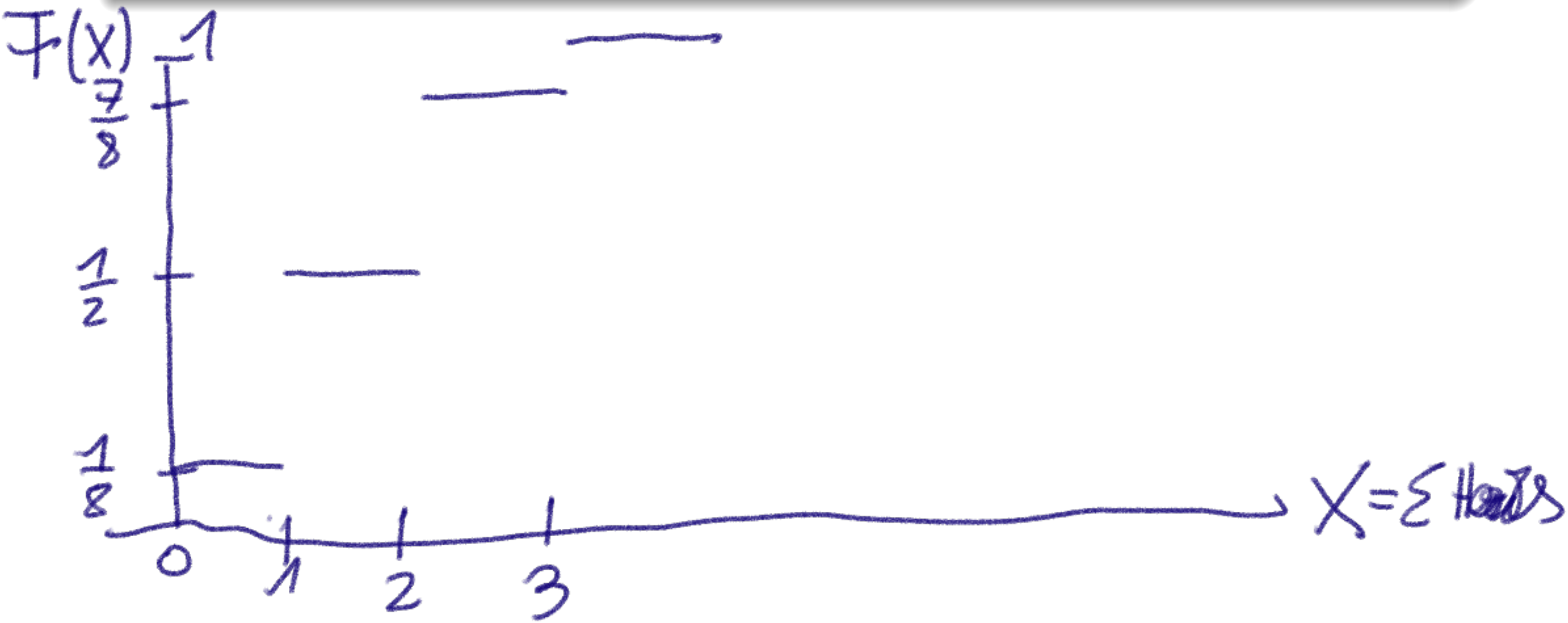
$$\begin{aligned} P(X>2) &= 1 - P(X \leq 2) \\ &= 1 - P(X=0) - P(X=1) - P(X=2) \\ &= 1 - e^{-\lambda} - e^{-\lambda}\lambda - e^{-\lambda}\frac{\lambda^2}{2} \end{aligned}$$

Cumulative distribution function

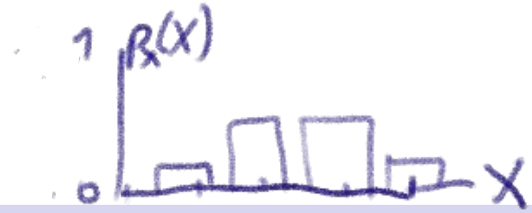
Definition

The *cumulative distribution function (cdf)* of a random variable X:

$$F : \mathbb{R} \to [0, 1], \qquad x \mapsto F(x) = \mathbf{P}(X \leq x).$$



Cumulative distribution function



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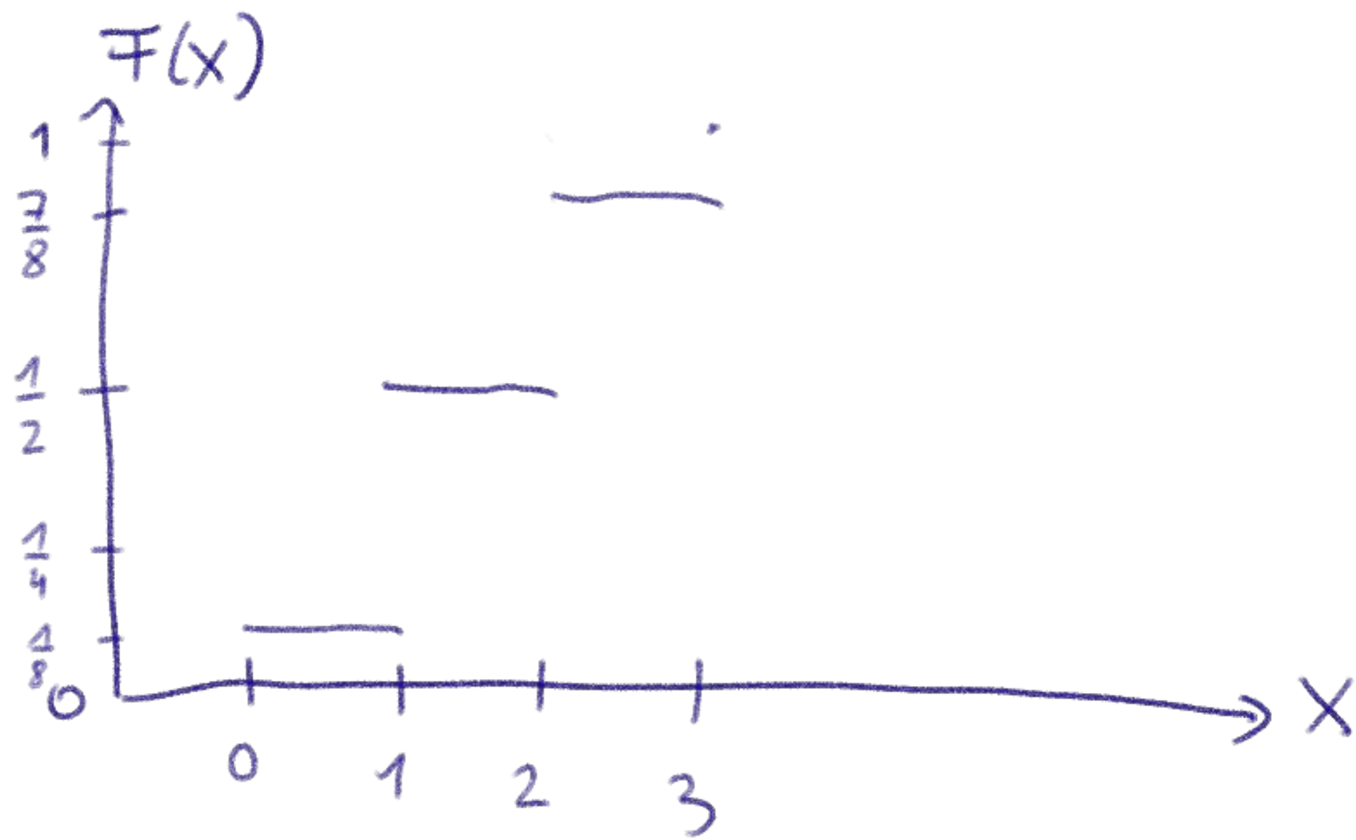

Cumulative distribution function

Will be very important when working with continuous random variables.

Proposition

A cumulative distribution function F:

- *is non-decreasing: if $x \leq y$ then $F(x) \leq F(y)$*
- *has limit $\lim_{x \to -\infty} F(x) = 0$ on the left*
- *has limit $\lim_{x \to \infty} F(x) = 1$ on the right*

Expectation

Expectation

Once we have a random variable, we want to quantify its *typical* behaviour in some sense. Two of the most often used quantities for this are the *expectation* and the *variance*.

Definition

The *expectation* of a discrete random variable X is:

$$\mathbf{E}X = \sum_i x_i \cdot \mathfrak{p}(x_i)$$

provided the sum exists. Also called *mean*, or *expected value*.

weighted average / center of mass

Examples

X = number on ~~a~~ fair die

$$\mathbb{E}X = \sum_i x_i \cdot p(x_i) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} \cdots 6 \frac{1}{6} = \frac{7}{2} = 3.5$$

$$\frac{n \cdot (n+1)}{2} \quad \frac{\cancel{6} \cdot 7}{2 \cdot \cancel{6}}$$

Examples

X = nr on fair die after roll

$$EX = \sum_{i=1}^{6} x_i \cdot p(X=i) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} \dots + 6\frac{1}{6} = \frac{7}{2}$$

expectation need not be a possible value

Properties of expectation

Proposition (expectation of a function of a random variable)

If X is a discrete random variable, and $g : \mathbb{R} \to \mathbb{R}$ a function, then:

$$\mathbf{E}g(X) = \sum_i g(x_i) \cdot \mathfrak{p}(x_i) \qquad \textit{(if it exists)}$$

Corollary (expectation is linear)

If X is a discrete random variable, and a, b fixed real numbers:

$$\mathbf{E}(aX + b) = a \cdot \mathbf{E}X + b.$$

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Corollary (expectation is linear)

If X is a discrete random variable, and a, b fixed real numbers:

$$\mathbf{E}(aX + b) = a \cdot \mathbf{E}X + b.$$

$$\sum_i c \cdot x_i \cdot p(x_i) = c \cdot \sum_i x_i \cdot p(x_i) \qquad \sum_i c \cdot p(x_i) = c$$

Moments

Definition (moments)

Let $n \in \mathbb{N}$. The n^{th} *moment* of a random variable X is:

$$\mathbf{E}X^n$$

The n^{th} *absolute moment* of X is:

$$\mathbf{E}|X|^n$$

$$\mathbb{E}X^n = \mathbb{E}(X^n) \neq (\mathbb{E}X)^n \quad !$$

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Variance

3. Variance

Definition (variance, standard deviation)

The *variance* and the *standard deviation* of a random variable are:

- $\mathbf{Var}\, X = \mathbf{E}(X - \mathbf{E}X)^2$.
- $\mathrm{SD}\, X = \sqrt{\mathbf{Var}\, X}$.

$$\mathrm{Var}\, X = \mathbf{E}(X - \mathbf{E}X)^2 = \mathbf{E}(X^2 - 2X\cdot\mathbf{E}X + \mathbf{E}X^2)$$

$$= \mathbf{E}(X^2) - \cancel{2}\mathbf{E}(X)\mathbf{E}X + \cancel{\mathbf{E}X^2})$$

$$= \mathbf{E}X^2 - (\mathbf{E}X)^2$$

3. Variance

Definition (variance, standard deviation)

The *variance* and the *standard deviation* of a random variable are:

- $\mathbf{Var}\,X = \mathbf{E}(X - \mathbf{E}X)^2$.
- $\mathrm{SD}\,X = \sqrt{\mathbf{Var}\,X}$.

$$\mathbb{V}\mathrm{ar}\,X = \mathbb{E}(X - \mathbb{E}X)^2 = \mathbb{E}X^2 - (\mathbb{E}X)^2$$

$$\mathbb{E}\left(X^2 - 2X \cdot \mathbb{E}X + (\mathbb{E}X)^2\right) = \mathbb{E}(X^2) - 2 \cdot \mathbb{E}X^2 \cdot \cancel{\mathbb{E}X} + (\mathbb{E}X)^2$$

$$= \mathbb{E}X^2 - (\mathbb{E}X)^2$$

Properties of the variance

Proposition (equivalent form of the variance)

$\mathbf{Var}\, X = \mathbf{E} X^2 - (\mathbf{E} X)^2$ *for any random variable* X.

Corollary

$\mathbf{E} X^2 \geq (\mathbf{E} X)^2$ *for any random variable* X,
with equality only if X *is constant.*

Examples

X = roll fair die

$$\mathbb{E}X = \frac{7}{2}$$

$$\operatorname{Var} X = \mathbb{E}X^2 - (\mathbb{E}X)^2$$

$$= 1 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 9 \cdot \frac{1}{6} \dots 36 \frac{1}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}$$

$$\operatorname{SD} X = \sqrt{\frac{35}{12}} \approx 1.71$$

two most important numbers about a fair die: average 3.5

standard deviation 1.7

Topics

- Bernoulli distribution: single trial
- Binomial distribution: many independent trials
- Poisson distribution: counting independent trials
- Geometric distribution: first success in independent trials

Bernoulli and Binomial distributions

~~Bernoulli~~ distribution

Binomial

Definition

Suppose that n independent trials are performed, each succeeding with probability p. Let X count the number of successes within the n trials. Then X has the *Binomial distribution with parameters n and p* or, in short, $X \sim Binom(n, p)$.

Special case $n = 1$ is called *Bernoulli distribution with parameter p*.

~~Bernoulli~~: mass function

Binomial

Proposition

Let $X \sim Binom(n,\, p)$. *Then* $X = 0,\, 1,\, \dots,\, n$, *and its mass function is*

$$\mathfrak{p}(i) = \mathbf{P}(X = i) = \binom{n}{i} p^i (1-p)^{n-i}, \qquad i = 0,\, 1,\, \dots,\, n.$$

In particular, the $Bernoulli(p)$ *variable can take on values* 0 *or* 1, *with respective probabilities*

$$\mathfrak{p}(0) = 1 - p, \qquad \mathfrak{p}(1) = p.$$

Mass function

$$p(i) = P(X=i) = \binom{n}{i} p^i (1-p)^{n-i} \qquad i=0,1,\dots,n.$$

1) $p(i) \geq 0$

2) $\sum_{i=0}^{n} p(i) = \sum \binom{n}{i} p^i \cdot (1-p)^{n-i} = (\cancel{p}+(1-\cancel{p}))^n = 1$

Newtons binomial formula

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} \cdot \underset{\substack{\parallel \\ p}}{a^i} \cdot \underset{\substack{\parallel \\ 1-p}}{b^{n-i}}$$

Mass function

$$p(i) = P(X=i) = \binom{n}{i} p^i (1-p)^{n-i} \qquad i = 0, 1, \dots, n.$$

$$p(i) \geq 0$$

$$\sum_{i=0}^{n} p(i) = \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} = (p + (1-p))^n = 1$$

Newton binomial formula:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k \cdot y^{n-k}$$

Example

Screws sold in packs of 10

Broken screws 0.1

If more than one defective screw then pack can be returned

$X \sim \mathrm{Binom}(10, 0.1)$

$$P(X>1) = 1 - P(X=0) - P(X=1)$$

$$- \binom{n}{0} \cdot 0.1^0 \cdot 0.9^{10} - \binom{n}{1} \cdot 0.1^1 \cdot 0.9^9 = .26$$

Example

Screws are sold in packs of 10.

Each one is broken with probability 0.1.

If more than one screw is broken, you can return pack.

what percentage of packs is returned

X = nr defective screws

$X \sim \text{Binom}(10, 0.1)$

$$P(X \geq 2) = 1 - P(X=0) - P(X=1)$$
$$= 1 - \binom{10}{0} 0.1^0 \cdot 0.9^{10} - \binom{10}{1} 0.1^1 \cdot 0.9^9 \approx 26\%$$

0.2639

~~Bernoulli~~ Binomial: expectation, variance

Proposition

Let $X \sim Binom(n, p)$. *Then:*

$$\mathbf{E}X = np, \qquad \textit{and} \qquad \mathbf{Var}\,X = np(1-p)$$

~~Bernoulli~~ Binomial: expectation, variance

Proposition

Let $X \sim Binom(n, p)$. *Then:*

$$\mathbf{E}X = np, \qquad \textit{and} \qquad \mathbf{Var}\, X = np(1-p)$$

Proof: $\mathbb{E}X = \sum_i i \cdot p(i) = \sum_i i \cdot \binom{n}{i} p^i (1-p)^{n-i}$

$$= \sum_{i=0}^{n} \binom{n}{i} \frac{d}{dt} t^i \Big|_{t=1} p^i (1-p)^{n-i}$$

$$= \frac{d}{dt} \left(\sum_{i=0}^{n} \binom{n}{i} t^i p^i (1-p)^{n-i} \right) \Big|_{t=1}$$

$$= \frac{d}{dt} \left((tp+1-p)^{n-1} \right|_{t=1}$$

$$= n \cdot (tp+1-p)^{n-1} \Big|_{t=1}$$

$$= np$$

Trick: $i = \frac{d}{dt} t^i \Big|_{t=1}$

Proof

$$\sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} = 1$$

$$x! = x\underbrace{(x-1)!}_{(x-1)!}$$

$$E[X] = \sum_{x=1}^{n} x \cdot \binom{n}{x} p^x (1-p)^{n-x} = \sum_{x=1}^{n} \frac{x\, n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$= \sum_{x=1}^{n} \frac{n!}{(x-1)!(n-x)!} p^x (1-p)^{n-x}$$

$$n - x = n - (x-1) - 1 = (n-1) - (x-1)$$

$$= \sum_{x=1}^{n} \frac{n(n-1)!}{(x-1)!((n-1)-(x-1))!} p^{(x-1)} p (1-p)^{(n-1)-(x-1)} = np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!((n-1)-(x-1))!} p^{(x-1)} (1-p)^{(n-1)-(x-1)}$$

$$z = x - 1, \quad m = n - 1$$

$$= np \sum_{z=0}^{n-1} \frac{(n-1)!}{z!((n-1)-z)!} p^z (1-p)^{(n-1)-z}$$

$$= np \sum_{z=0}^{n} \frac{m!}{z!(m-z)!} p^z (1-p)^{m-z} = np \underbrace{\sum_{z=0}^{m} \binom{m}{z} p^z (1-p)^{m-z}}_{1} = np$$

https://www.youtube.com/watch?v=s4uPtdnUTfc&t=663s

Poisson distribution

Poisson: mass function

The Poisson distribution is of central importance in Probability.
Will later see relation to Binomial.

Definition

Fix a positive real number λ. The random variable X is *Poisson distributed with parameter* λ, in short $X \sim Poi(\lambda)$, if it is non-negative integer valued, and its mass function is

$$\mathfrak{p}(i) = \mathbf{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}, \qquad i = 0, 1, 2, \ldots$$

is a p.m.f

but why this function

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$$\mathfrak{p}(i) = \mathbf{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}, \qquad i = 0, 1, 2, \ldots$$

Indeed a p.m.f. used Taylor series to prove this.

Fine, but why this distribution?

Poisson approximation of Binomial

Proposition

Fix $\lambda > 0$, and suppose that $Y_n \sim Binom(n, p)$ with $p = p(n)$ in such a way that $n \cdot p \to \lambda$. Then the distribution of Y_n converges to $Poisson(\lambda)$:

$$\forall i \geq 0 \qquad \mathbf{P}(Y_n = i) \underset{n \to \infty}{\longrightarrow} e^{-\lambda} \frac{\lambda^i}{i!}.$$

Poisson approximation of Binomial

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$$\forall i \geq 0 \qquad \mathbf{P}(Y_n = i) \underset{n\to\infty}{\longrightarrow} e^{-\lambda}\frac{\lambda^i}{i!}.$$

Take $Y_n = Binom(n,p)$ with large n, small p, such that $np = \lambda$ then Y_n is approximately Poisson distributed with parameter λ.

Proof

$$P(Y_n = i) \longrightarrow e^{-\lambda} \frac{\lambda^i}{i!}$$

Proof

$$P(Y_n = i) \xrightarrow[n\to\infty]{} e^{-\lambda} \frac{\lambda^i}{i!}$$

$$P(Y_n = i) = \binom{n}{i} p^i (1-p)^{n-i}$$

$$= \frac{1}{i!} \cdot np \cdot (n-1)p \cdot (n-2)p \cdots (n-i+1)p \, \frac{(1-p)^n}{(1-p)^i}$$

$$\xrightarrow[n\to\infty]{} \frac{1}{i!} \cdot \lambda \cdot \lambda \cdot \lambda \cdots \lambda \cdot e^{-\lambda}$$

$$= \frac{1}{i!} \cdot \lambda^i \cdot e^{-\lambda} = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

$$np \longrightarrow \lambda$$

$$(n-1)p \longrightarrow \lambda$$

$$(n-2)p \longrightarrow \lambda$$

$$(1-p)^n \longrightarrow \left(1 - \frac{\lambda}{n}\right)^n \longrightarrow e^{-\lambda}$$

$$(1-p)^i \longrightarrow 1$$

Poisson: expectation, variance

Proposition

For $X \sim Poi(\lambda)$, $\mathbf{E}X = \mathbf{Var}\,X = \lambda$.

Example $X \sim \mathrm{Poi}(\lambda)$ $\mathbb{E}X = \mathrm{Var}\, X = \lambda$

Book has average $\frac{1}{2}$ typo per page.

Chance that a random page has ≥ 3 typos?

$$X \sim \mathrm{Poisson}(\lambda)$$

$$\mathbb{E}X = \lambda = \frac{1}{2}$$

$$P(X \geq 3) = 1 - P(X \leq 2)$$

$$1 - P(X=0) - P(X=1) - P(X=2)$$

$$1 - e^{-\frac{1}{2}} \cdot \frac{\frac{1}{2}^0}{0!} - e^{-\frac{1}{2}} \frac{\frac{1}{2}^1}{1!} - e^{-\frac{1}{2}} \frac{\frac{1}{2}^2}{2!} \doteq 1.4\%$$

Example $X \sim \mathrm{Poi}(\lambda)$ $\mathbb{E}X = \mathrm{Var}\, X = \lambda$

Book has average $\frac{1}{2}$ typo per page.

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$$
\begin{aligned}
P(X \geq 3) &= 1 - P(X \leq 2) \\
&= 1 - P(X=0) - P(X=1) - P(X=2) \\
&= 1 - e^{-\frac{1}{2}} \cdot \frac{\left(\frac{1}{2}\right)^0}{0!} - e^{-\frac{1}{2}} \cdot \frac{\left(\frac{1}{2}\right)^1}{1!} - e^{-\frac{1}{2}} \cdot \frac{\left(\frac{1}{2}\right)^2}{2!} \approx 1.4\%
\end{aligned}
$$

Example $X \sim \mathrm{Poi}(\lambda)$ $\mathbb{E}X = \mathrm{Var}\,X = \lambda$

Book has average $\frac{1}{2}$ typo per page.
Chance that a random page has ≥ 3 typos?

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$$= 1 - e^{-\frac{1}{2}} \cdot \frac{\left(\frac{1}{2}\right)^0}{0!} - e^{-\frac{1}{2}} \cdot \frac{\left(\frac{1}{2}\right)^1}{1!} - e^{-\frac{1}{2}} \cdot \frac{\left(\frac{1}{2}\right)^2}{2!} \approx 1.4\%$$

Poisson distribution models: many independent trials with small success probability summing up to "a few" in expectation

e.g.
- nr of typos on page of book
- nr of citizens over 100 years in big city
- nr of calls per hour in customer center

Examples

Examples

Defective screws.

$X \sim \text{Binom}(10, 0.1)$

Approximated by Poisson(1) as $\lambda = 10 \cdot 0.1 = 1 = np$

$$P(X \geq 2) = 1 - P(X=0) - P(X=1)$$
$$= 1 - e^{-1}\frac{1^0}{0!} - e^{-1}\frac{1^1}{1!} \approx 26\%$$

0.26424

Poisson: History 1

Poisson: History 2

Geometric distribution

Geometric: mass function

Again independent trials, but now ask: when is the first success?

Definition

Suppose that independent trials, each succeeding with probability p, are repeated until the first success. The total number X of trials made has the *Geometric*(p) distribution (in short, $X \sim \mathit{Geom}(p)$).

Proposition

X can take on positive integers, with probabilities
$\mathfrak{p}(i) = (1-p)^{i-1} \cdot p$, $i = 1, 2, \ldots$.

all these numbers are positive and sum up to 1.

Geometric: mass function

Corollary

The Geometric random variable is (discrete) memoryless:

$$\mathbf{P}\{X \geq n + k \mid X > n\} = \mathbf{P}\{X \geq k\}$$

for every $k \geq 1$, $n \geq 0$.

Geometric: expectation, variance

Proposition two most important numbers

For a Geometric(p) random variable X:

$$\mathbf{E}X = \frac{1}{p} \qquad \mathbf{Var}\,X = \frac{1-p}{p^2}$$

Example

$\mathbb{E}X = \frac{1}{p}$ $\quad \mathrm{Var}\, X = \frac{1-p}{p^2}$

first 3 on fair die $\quad X \sim \mathrm{Geom}\left(\frac{1}{6}\right)$

$$\mathbb{E}X = \frac{1}{\frac{1}{6}} = 6$$

$$SD(X) = \sqrt{\mathrm{Var}\, X} = \sqrt{\frac{1-\frac{1}{6}}{\frac{1}{36}}} = \sqrt{30} = 5.48$$

Chance 3 comes on the 7th roll is

$$P(X=7) = \left(1-\frac{1}{6}\right)^6 \cdot \frac{1}{6} = 5.6\%$$

Example

$\mathbb{E}X = \frac{1}{p} \quad \mathrm{Var}\,X = \frac{1-p}{p^2}$

first 3 on fair die takes $X \sim \mathrm{Geom}(\frac{1}{6})$ rolls

$$\mathbb{E}X = \frac{1}{1/6} = 6$$

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Chance that first 3 comes on 7th roll is:

$$P(X=7) = \left(1-\frac{1}{6}\right)^6 \cdot \frac{1}{6} = 5.6\%$$

Summary

- Bernoulli distribution: single trial
- Binomial distribution: many independent trials
- Poisson distribution: counting independent trials
- Geometric distribution: first success in independent trials