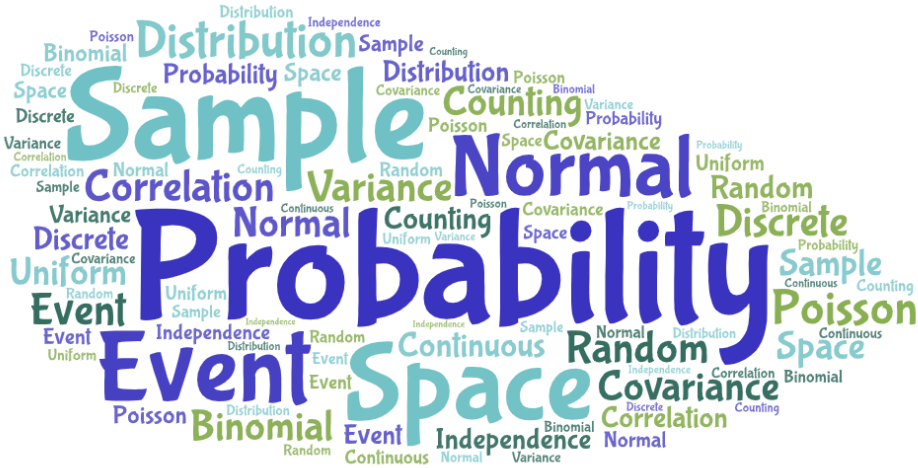
Discrete Mathematics and Probability

Week 8



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Topics

- Bernoulli distribution: single trial
- Binomial distribution: many independent trials
- Poisson distribution: counting independent trials
- Geometric distribution: first success in independent trials

Bernoulli and Binomial distributions

Bernoulli distribution

Definition

Suppose that n independent trials are performed, each succeeding with probability p. Let X count the number of successes within the n trials. Then X has the *Binomial distribution with parameters* n *and* p or, in short, $X \sim Binom(n, p)$.

Special case $n = 1$ is called *Bernoulli distribution with parameter* p.

Bernoulli: mass function

Proposition

Let $X \sim Binom(n, p)$. Then $X = 0, 1, \ldots, n$, and its mass function is

$$\mathfrak{p}(i) = \mathbf{P}(X = i) = \binom{n}{i} p^i (1-p)^{n-i}, \qquad i = 0, 1, \ldots, n.$$

In particular, the $Bernoulli(p)$ variable can take on values 0 or 1, with respective probabilities

$$\mathfrak{p}(0) = 1 - p, \qquad \mathfrak{p}(1) = p.$$

Mass function

Example

Bernoulli: expectation, variance

Proposition

Let $X \sim Binom(n, p)$. *Then:*

$$\mathbf{E}X = np, \qquad and \qquad \mathbf{Var}\, X = np(1-p)$$

Proof

Poisson distribution

Poisson: mass function

The Poisson distribution is of central importance in Probability.
Will later see relation to Binomial.

Definition

Fix a positive real number λ. The random variable X is *Poisson distributed with parameter* λ, in short $X \sim Poi(\lambda)$, if it is non-negative integer valued, and its mass function is

$$\mathfrak{p}(i) = \mathbf{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}, \qquad i = 0, 1, 2, \ldots$$

Poisson approximation of Binomial

Proposition

Fix $\lambda > 0$, and suppose that $Y_n \sim Binom(n, p)$ with $p = p(n)$ in such a way that $n \cdot p \to \lambda$. Then the distribution of Y_n converges to $Poisson(\lambda)$:

$$\forall i \geq 0 \qquad \mathbf{P}(Y_n = i) \underset{n\to\infty}{\longrightarrow} e^{-\lambda}\frac{\lambda^i}{i!}.$$

Proof

Poisson: expectation, variance

Proposition

For $X \sim Poi(\lambda)$, $\mathbf{E}X = \mathbf{Var}\, X = \lambda$.

Example

Examples

Geometric distribution

Geometric: mass function

Again independent trials, but now ask: when is the first success?

Definition

Suppose that independent trials, each succeeding with probability p, are repeated until the first success. The total number X of trials made has the *Geometric*(p) distribution (in short, $X \sim Geom(p)$).

Proposition

X can take on positive integers, with probabilities $\mathfrak{p}(i) = (1-p)^{i-1} \cdot p$, $i = 1, 2, \ldots$.

Geometric: mass function

Corollary

The Geometric random variable is (discrete) memoryless:

$$\mathbf{P}\{X \geq n + k \,|\, X > n\} = \mathbf{P}\{X \geq k\}$$

for every $k \geq 1$, $n \geq 0$.

Geometric: expectation, variance

Proposition

For a Geometric(p) random variable X:

$$\mathbf{E}X = \frac{1}{p} \qquad\qquad \mathbf{Var}\,X = \frac{1-p}{p^2}$$

Example

Summary

- Bernoulli distribution: single trial
- Binomial distribution: many independent trials
- Poisson distribution: counting independent trials
- Geometric distribution: first success in independent trials