Retrieve your submissions from Homework 1 in Week 2 as well as the solution notes on the course website. Compare solutions around the group. Ask your tutor for help if there is anything you do not understand.

What counterexample did you pick in Question 1(b)? Can you find others? Do all counterexamples n have $(n^2 + 4)$ a multiple of 5?

Now work together as a group on each of the following tasks.

Task A

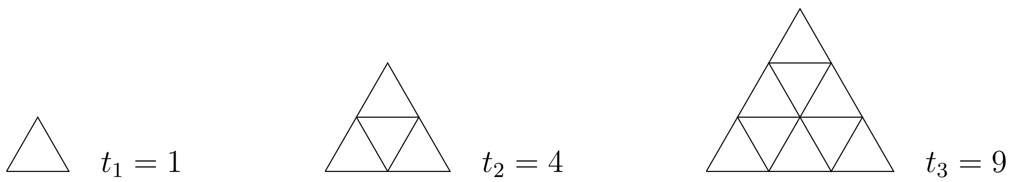
Below are two constructions of numerical sequences. For each one carry out the following steps.

- **(a)** Extend the sequence given for $n = 1, 2, 3, 4$, and 5.
- **(b)** Write down a conjecture for what you think the nth term of the sequence would be.
- **(c)** Prove your conjecture correct or find a counterexample.

For both of these examples there is an obvious conjecture for how the sequence continues: one of those "obvious" conjectures is right and one is wrong.

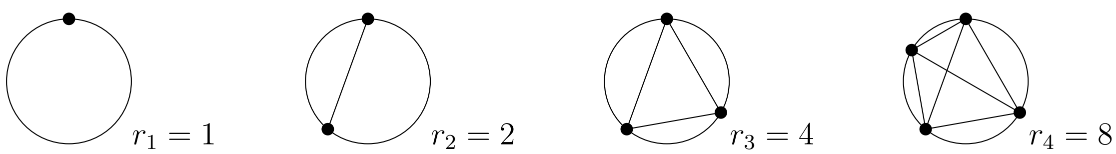
Triangulation

The sequence $t_1, t_2, \ldots, t_n, \ldots$ counts how many small triangles make a big triangle with n to a side, as shown below.



Circle Subdivision

The sequence $r_1, r_2, \ldots, r_n, \ldots$ counts the largest number of regions a circle can be divided into by straight lines joining n points on the circumference, as shown below.



Task B

Use mathematical induction to prove that for any integer $n \geq 0$, $7^n - 2^n$ is divisible by 5.

Task C

Suppose that $f_0, f_1, f_2, \ldots$ is a sequence defined as follows:

$$f_0 = 5 \qquad f_1 = 16 \qquad f_k = 7f_{(k-1)} - 10f_{(k-2)} \quad \text{for any integer } k \geq 2.$$

Prove by mathematical induction that $f_n = 3 \cdot 2^n + 2 \cdot 5^n$ for all non-negative integers n.

This is Question 6 from Exercise Set 5.4 in the Epp textbook. Does your solution use strong induction or not? How do you tell?

Task D

Sequence $c_0, c_1, c_2, \ldots$ is defined below.

$$c_0 = 2 \quad c_1 = 2 \quad c_2 = 6$$
$$c_k = 3c_{k-3} \quad \text{for every integer } k > 2.$$

Prove that all elements of the sequence are even.

Task E

Calculate 9^k for $k = 0, 1, 2, 3, 4$, and 5. Use these results to make a conjecture relating the parity of n to the units digit of 9^n for non-negative integers n.

Use mathematical induction to prove your conjecture.

Solution Notes

For Homework 1 see the solution notes on the course web pages.

For Question 1(b) there is an infinite supply of counterexamples: for example, if $n = (10k + 1)$ or $n = (10k - 1)$ for any integer k then $(n^2 + 4)$ is divisible by 5.

$$(10k \pm 1)^2 + 1 \;=\; 100k^2 \pm 20k + 1 + 4 \;=\; 5 \cdot (20k \pm 4k + 1)$$

Similar reasoning shows that $n = (26k \pm 3)$ then $(n^2 + 4)$ is a multiple of 13, giving infinitely more counterexamples not divisible by 5.

These infinities of counterexamples of all kinds are in fact to be expected: formulas that actually do generate primes are extremely hard to find and much more complex than $(n^2 + 4)$. See the Wikipedia page on a *Formula for Primes* to learn more about this.
https://en.wikipedia.org/wiki/Formula_for_primes

Task A

Triangulation

The first five values of the sequence are $1, 4, 9, 16$, and 25, suggesting $t_n = n^2$.

This turns out to be correct and can be proved in several ways. For example, by mathematical induction on the recurrence relation $t_{n+1} = t_n + n + (n + 1)$, noting that expanding the triangle of side n can be done by adding on the bottom two rows of triangles: one row of n pointing down and another of $(n + 1)$ pointing up. Or, by calculating the area of the large triangle as $\frac{\sqrt{3}}{4}n^2$ and of the small triangles as $\frac{\sqrt{3}}{4}$. Or even just remembering that area increases as the square of length.

Circle Subdivision

The first five values of the sequence are $1, 2, 4, 8$, and 16, suggesting $r_n = 2^{n-1}$.

This is incorrect: the next value $r_6 = 31$ as can be discovered by drawing out the diagram. The actual formula for r_n is a quartic polynomial:

$$r_n \;=\; \frac{n}{24}n^3 - 6n^2 + 23n - 18 + 1 \;=\; \binom{n}{4} + \binom{n}{2} + 1 \,.$$

For more discussion, including proofs of this formula, see Wikipedia on *Dividing a Circle into Areas.*
https://en.wikipedia.org/wiki/Dividing_a_circle_into_areas.

Task B

This and the following notes are taken from the Instructor's Manual for the Epp textbook.

Proof (by mathematical induction): Let the property $P(n)$ be the sentence

$$7^n - 2^n \text{ is divisible by } 5. \qquad \leftarrow P(n)$$

We will prove that $P(n)$ is true for every integer $n \geq 0$.

Show that $P(0)$ is true: $P(0)$ is true because $7^0 - 2^0 = 1 - 1 = 0$ and 0 is divisible by 5 (since $0 = 5 \cdot 0$).

Show that for every integer $k \geq 0$, if $P(k)$ is true then $P(k+1)$ is true:

Let k be any integer with $k \geq 0$, and suppose

$$7^k - 2^k \text{ is divisible by } 5. \qquad \leftarrow \begin{array}{l} P(k) \\ \text{inductive hypothesis} \end{array}$$

We must show that

$$7^{k+1} - 2^{k+1} \text{ is divisible by } 5. \qquad \leftarrow P(k+1)$$

By definition of divisibility, the inductive hypothesis is equivalent to the statement $7^k - 2^k = 5r$ for some integer r. Then

$$\begin{aligned} 7^{k+1} - 2^{k+1} &= 7 \cdot 7^k - 2 \cdot 2^k \\ &= (5+2) \cdot 7^k - 2 \cdot 2^k \\ &= 5 \cdot 7^k + 2 \cdot 7^k - 2 \cdot 2^k \\ &= 5 \cdot 7^k + 2(7^k - 2^k) && \text{by algebra} \\ &= 5 \cdot 7^k + 2 \cdot 5r && \text{by inductive hypothesis} \\ &= 5(7^k + 2r) && \text{by algebra.} \end{aligned}$$

Now $7^k + 2r$ is an integer because products and sums of integers are integers. Therefore, by definition of divisibility, $7^{k+1} - 2^{k+1}$ is divisible by 5 *[as was to be shown]*.

Task C

Proof (by strong mathematical induction): Let the property $P(n)$ be the equation

$$f_n = 3\cdot 2^n + 2\cdot 5^n.$$

We will prove that $P(n)$ is true for every integer $n \geq 0$.

Show that $P(0)$ and $P(1)$ are true: By definition of $f_0, f_1, f_2, \ldots$, we have that $f_0 = 5$ and $f_1 = 16$. Since $3\cdot 2^0 + 2\cdot 5^0 = 3 + 2 = 5$ and $3\cdot 2^1 + 2\cdot 5^1 = 6 + 10 = 16$, both $P(0)$ and $P(1)$ are true.

Show that for every integer $k \geq 1$, if $P(i)$ is true for each integer i from 0 through k, then $P(k+1)$ is true: Let k be any integer with $k \geq 1$, and suppose

$$f_i = 3\cdot 2^i + 2\cdot 5^i \text{ for every integer } i \text{ with } 0 \leq i \leq k. \qquad \leftarrow \text{inductive hypothesis}$$

We must show that

$$f_{k+1} = 3\cdot 2^{k+1} + 2\cdot 5^{k+1}.$$

Now

$$\begin{aligned}
f_{k+1} &= 7f_k - 10f_{k-1} && \text{by definition of } f_0, f_1, f_2, \ldots \\
&= 7(3\cdot 2^k + 2\cdot 5^k) - 10(3\cdot 2^{k-1} + 2\cdot 5^{k-1}) && \text{by inductive hypothesis} \\
&= 7(6\cdot 2^{k-1} + 10\cdot 5^{k-1}) - 10(3\cdot 2^{k-1} + 2\cdot 5^{k-1}) && \text{since } 2^k = 2\cdot 2^{k-1} \text{ and } 5^k = 5\cdot 5^{k-1} \\
&= (42\cdot 2^{k-1} + 70\cdot 5^{k-1}) - (30\cdot 2^{k-1} + 20\cdot 5^{k-1}) \\
&= (42-30)\cdot 2^{k-1} + (70-20)\cdot 5^{k-1} \\
&= 12\cdot 2^{k-1} + 50\cdot 5^{k-1} \\
&= 3\cdot 2^2\cdot 2^{k-1} + 2\cdot 5^2\cdot 5^{k-1} \\
&= 3\cdot 2^{k+1} + 2\cdot 5^{k+1} && \text{by algebra,}
\end{aligned}$$

[as was to be shown].

[Since both the basis and the inductive steps have been proved, we conclude that $P(n)$ is true for every integer $n \geq 0$.]

Task D

Proof (by strong mathematical induction): Let the property $P(n)$ be the sentence

$$c_n \text{ is even.}$$

We will prove that $P(n)$ is true for every integer $n \geq 0$.

Show that $P(0)$, $P(1)$, and $P(2)$ are true: By definition of $c_0, c_1, c_2, \ldots$, we have that $c_0 = 2$, $c_1 = 2$, and $c_2 = 6$ and 2, 2, and 6 are all even. So $P(0)$, $P(1)$, and $P(2)$ are all true.

Show that for every integer $k \geq 2$, if $P(i)$ is true for each integer i from 0 through k, then $P(k+1)$ is true: Let k be any integer with $k \geq 2$, and suppose

$$c_i \text{ is even for every integer } i \text{ with } 0 \leq i \leq k \qquad \leftarrow \text{inductive hypothesis}$$

We must show that

$$c_{k+1} \text{ is even.}$$

Now by definition of $c_0, c_1, c_2, \ldots, c_{k+1} = 3c_{k-2}$. Since $k \geq 2$, we have that $0 \leq k-2 \leq k$, and so, by inductive hypothesis, c_{k-2} is even. Now the product of an even integer with any integer is even *[properties 1 and 4 of Example 4.2.3]*, and hence $3c_{k-2}$, which equals c_{k+1}, is also even *[as was to be shown]*.

[Since both the basis and the inductive steps have been proved, we conclude that $P(n)$ is true for every integer $n \geq 0$.]

Task E

$9^1 = 9,\ 9^2 = 81,\ 9^3 = 729,\ 9^4 = 6561,$ and $9^5 = 59049.$

Conjecture: For every integer $n \geq 0$, the units digit of 9^n is 1 if n is even and is 9 if n is odd.

Proof (by mathematical induction): Let the property $P(n)$ be the sentence

The units digit of 9^n is 1 if n is even and is 9 if n is odd.

We will prove that $P(n)$ is true for every integer $n \geq 0$.

Show that $\boldsymbol{P(0)}$ ***is true***: $P(0)$ is true because 0 is even and the units digit of $9^0 = 1$.

Show that for every integer $\boldsymbol{k \geq 0}$, ***if*** $\boldsymbol{P(i)}$ ***is true for each integer*** $\boldsymbol{i}$ ***from*** $\boldsymbol{0}$ ***through*** $\boldsymbol{k}$, ***then*** $\boldsymbol{P(k+1)}$ ***is true***: Let k be any integer with $k \geq 0$, and suppose:

For every integer i from 0 through k,

$$\text{the units digit of } 9^i = \begin{cases} 1 \text{ if } i \text{ is even} \\ 9 \text{ if } i \text{ is odd} \end{cases} \quad \leftarrow \text{ inductive hypothesis}$$

We must show that

$$\text{the units digit of } 9^{k+1} = \begin{cases} 1 \text{ if } i \text{ is even} \\ 9 \text{ if } i \text{ is odd} \end{cases} \quad \leftarrow \quad P(k+1)$$

Case 1 (k+1 is even): In this case k is odd, and so, by inductive hypothesis, the units digit of 9^k is 9. This implies that there is an integer a so that $9^k = 10a + 9$, and hence

$$\begin{aligned} 9^{k+1} &= 9^1 \cdot 9^k && \text{by algebra (a law of exponents)} \\ &= 9(10a+9) && \text{by substitution} \\ &= 90a + 81 \\ &= 90a + 80 + 1 \\ &= 10(9a+8) + 1 && \text{by algebra.} \end{aligned}$$

Because $9a + 8$ is an integer, it follows that the units digit of 9^{k+1} is 1.

Case 2 $\boldsymbol{(k+1}$ ***is odd)***: In this case k is even, and so, by inductive hypothesis, the units digit of 9^{k-1} is 1. This implies that there is an integer a so that $9^k = 10a + 1$, and hence

$$\begin{aligned} 9^{k+1} &= 9^1 \cdot 9^k && \text{by algebra (a law of exponents)} \\ &= 9(10a+1) && \text{by substitution} \\ &= 90a + 9 \\ &= 10(9a) + 9 && \text{by algebra.} \end{aligned}$$

Because $9a$ is an integer, it follows that the units digit of 9^{k+1} is 9.

Hence in both cases the units digit of 9^{k+1} is as specified in $P(k+1)$ *[as was to be shown]*.