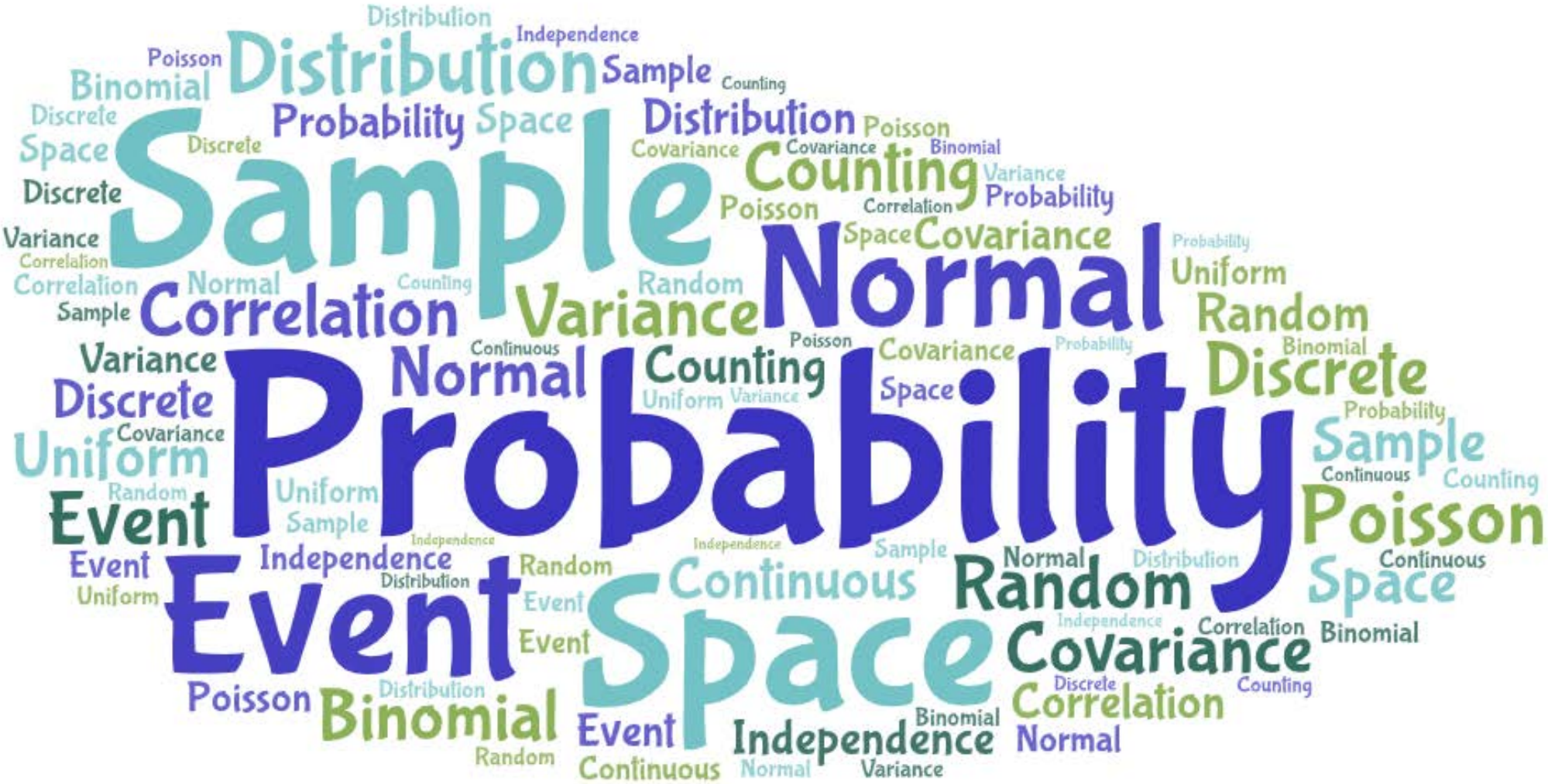
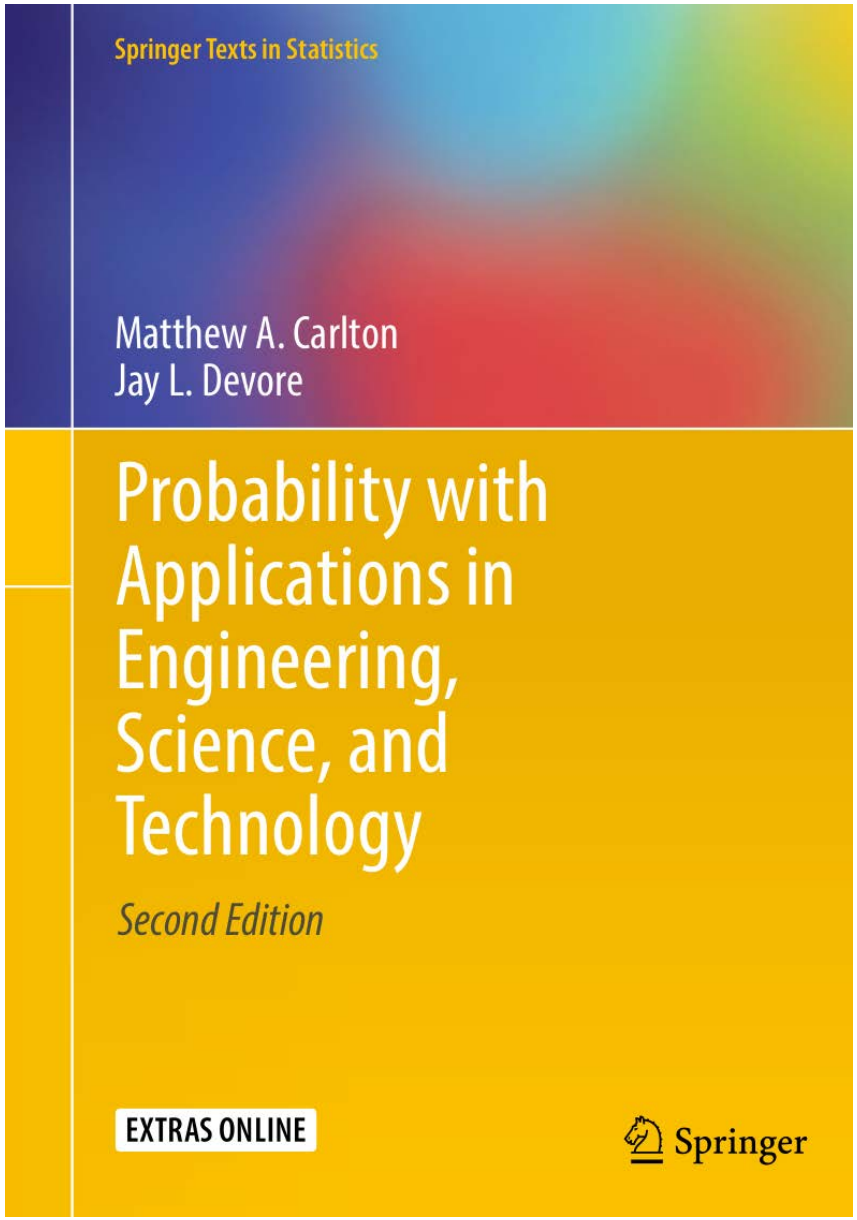
Discrete Mathematics and Probability

Week 7



MARKULF Chris Heunen KOHLWEISS


Springer Texts in Statistics
Matthew A. Carlton
Jay L. Devore
Probability with Applications in Engineering, Science, and Technology
Second Edition
EXTRAS ONLINE
Springer

Topics

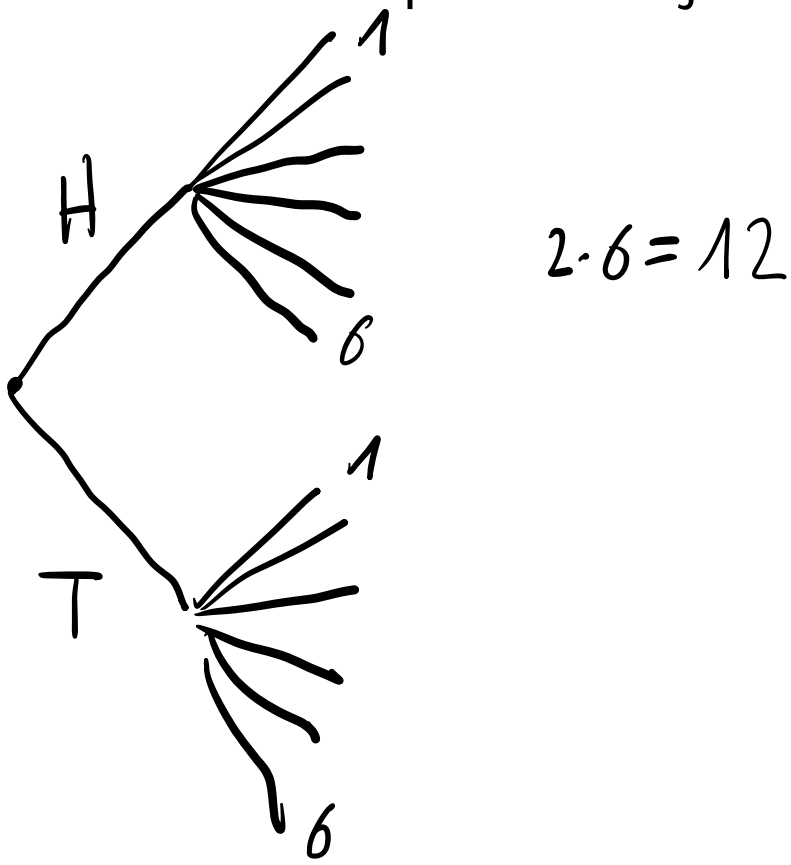
- Counting: thinking algorithmically
- Events: what could happen in principle
- Experiments: how can events interact
- Probability: quantifying what could happen

Counting

Counting

Basic principles of combinatorics:

- if an experiment has n outcomes;
  and another experiment has m outcomes,
- then the two experiments jointly have $n \cdot m$ outcomes.



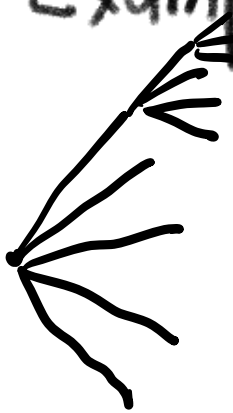
$2 \cdot 6 = 12$

Permutations

Definition

Let $H = \{h_1, h_2, \ldots, h_n\}$ be a set of n different objects. The *permutations* of H are the different orders in which you can write all of its elements.

Example: Horse race $\{A, B, C, D, E\}$



$\{E\,D\,A\,B\,C\}$

$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5!$

$n \cdot (n-1) \cdots 2 \cdot 1 = n!$

Permutations

Definition

Let $H = \{h_1, h_2, \ldots, h_n\}$ be a set of n different objects. The *permutations* of H are the different orders in which you can write all of its elements.

Example: Horse race $\{A, B, C, D, E\}$

$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ permutations

e.g. (E, C, B, A, D)

gold silver bronze

Permutations

Definition

Let $H = \{h_1, h_2, \ldots, h_n\}$ be a set of n different objects. The *permutations* of H are the different orders in which you can write all of its elements.

Example: Horse race $\{A, B, C, D, E\}$

$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ permutations

e.g. (E, C, B, A, D)

gold silver bronze

How many in general? $n! = n \cdot (n-1) \cdot (n-2) \cdots 1$

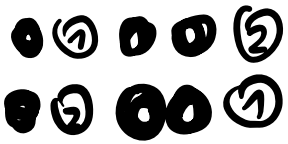
Permutations with repetitions

Definition

Let $H = \{h_1 \dots h_1,\ h_2 \dots h_2,\ \dots,\ h_r \dots h_r\}$ be a set of r different types of repeated objects: n_1 many of h_1, n_2 of h_2, $\dots$ n_r of h_r. The *permutations with repetitions* of H are the different orders in which you can write all of its elements.



$$5!$$



$$\frac{5!}{2!\ 3!}$$

$$\frac{\left(\sum_{i=1}^{r} n_i\right)!}{\prod_{i=1}^{r} n_i!}$$

Permutations with repetitions

class question

Definition

Let $H = \{h_1 \dots h_1,\ h_2 \dots h_2,\ \dots,\ h_r \dots h_r\}$ be a set of r different types of repeated objects: n_1 many of h_1, n_2 of h_2, $\dots$ n_r of h_r. The *permutations with repetitions* of H are the different orders in which you can write all of its elements.

not all elements are different

have a little think! Maybe more than a little think

Permutations with repetitions

Definition

Let $H = \{h_1 \dots h_1,\ h_2 \dots h_2,\ \dots,\ h_r \dots h_r\}$ be a set of r different types of repeated objects: n_1 many of h_1, n_2 of h_2, $\dots$ n_r of h_r. The *permutations with repetitions* of H are the different orders in which you can write all of its elements.

not all elements are different

have a little think! Maybe more than a little think

$$\frac{n!}{n_1! \cdots n_r!} \qquad n = n_1 + n_2 \cdots n_r$$

$$\prod_{i=1}^{r} n_r!$$

k-Permutations

Definition

Let $H = \{h_1, h_2, \ldots, h_n\}$ be a set of n different objects. The *k-permutations* of H are the different ways in which one can pick and write k of its elements of H in order.

How many?

(BDC)EA

$$\frac{5!}{2!} = \frac{120}{2} = 6$$

$$\frac{n!}{(n-k)!}$$

k-Permutations

Definition

Let $H = \{h_1, h_2, \ldots, h_n\}$ be a set of n different objects. The *k-permutations* of H are the different ways in which one can pick and write k of its elements of H in order.

How many?

Example: top 3 in horse race $\{A, B, C, D, E\}$

k-Permutations

Definition

Let $H = \{h_1, h_2, \ldots, h_n\}$ be a set of n different objects. The *k-permutations* of H are the different ways in which one can pick and write k of its elements of H in order.

How many?

Example: top 3 in horse race $\{A, B, C, D, E\}$

e.g. (E, B, A)

$$\frac{5!}{2!} = 60$$

k-Permutations

Definition

Let $H = \{h_1, h_2, \ldots, h_n\}$ be a set of n different objects. The *k-permutations* of H are the different ways in which one can pick and write k of its elements of H in order.

How many? $P_{n,k} = \frac{n!}{(n-k)!}$

Example: top 3 in horse race $\{A, B, C, D, E\}$

e.g. (E, B, A)

$$\frac{5!}{2!} = 60$$

k-Permutations with repetitions

Definition

Let $H = \{h_1 \ldots, h_2 \ldots, \ldots, h_r \ldots\}$ be a set of r different types of repeated objects, each of infinite supply. The *k-permutations with repetitions* of H are the different orders in which one can write an ordered sequence of length k using the elements of H.

Again set with repeated objects, but now each of infinite supply.

$26 \cdot 26 \cdot 26$ number of 3 letter words.

k-length bitstrings. $\underbrace{2 \cdot 2 \cdots 2}_{k\text{-times}} = 2^k$

k-Permutations with repetitions

Definition

Let $H = \{h_1 \ldots, h_2 \ldots, \ldots, h_r \ldots\}$ be a set of r different types of repeated objects, each of infinite supply. The *k-permutations with repetitions* of H are the different orders in which one can write an ordered sequence of length k using the elements of H.

Again set with repeated objects, but now each of infinite supply.

Example: # 3-letter words of letters English Alphabet $= 26^3 = 26 \cdot 26 \cdot 26$
$= 676$

k-Permutations with repetitions

Definition

Let $H = \{h_1 \ldots, h_2 \ldots, \ldots, h_r \ldots\}$ be a set of r different types of repeated objects, each of infinite supply. The *k-permutations with repetitions* of H are the different orders in which one can write an ordered sequence of length k using the elements of H.

Again set with repeated objects, but now each of infinite supply.

Example: # 3-letter words of letters English Alphabet $= 26^3 = 17576$

k-bit strings $= 2^k$

k-Combinations

Definition

Let $H = \{h_1, h_2, \ldots, h_n\}$ be a set of n different objects. The *k-combinations* of H are the different ways in which one can pick k of its elements without order.

$$C_{n,k} = \binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}$$

$$C_{26,3} = \binom{26}{3} = \frac{26!}{3!\,(26-3)!}$$

k-Combinations

Definition

Let $H = \{h_1, h_2, \ldots, h_n\}$ be a set of n different objects. The *k-combinations* of H are the different ways in which one can pick k of its elements without order.

How many?

k-Combinations

Definition

Let $H = \{h_1, h_2, \ldots, h_n\}$ be a set of n different objects. The *k-combinations* of H are the different ways in which one can pick k of its elements without order.

How many? $C_{n,k} = \binom{n}{k} = \frac{n!}{k!(n-k)!}$

"binomial coefficient" $C_{n,k} = \frac{P_{n,k}}{k!}$

k-Combinations

Definition

Let $H = \{h_1, h_2, \ldots, h_n\}$ be a set of n different objects. The *k-combinations* of H are the different ways in which one can pick k of its elements without order.

How many? $C_{n,k} = \binom{n}{k} = \frac{n!}{k!(n-k)!}$

"binomial coefficient" $C_{n,k} = \frac{P_{n,k}}{k!}$

Example: # ways to form committee of 5 students from class of 30

k-Combinations

Definition

Let $H = \{h_1, h_2, \ldots, h_n\}$ be a set of n different objects. The *k-combinations* of H are the different ways in which one can pick k of its elements without order.

How many? $C_{n,k} = \binom{n}{k} = \frac{n!}{k!(n-k)!}$

"binomial coefficient" $C_{n,k} = \frac{P_{n,k}}{k!}$

Example: # ways to form committee of 5 students from class of 30

$= \binom{30}{5} = \frac{30!}{6!25!} = 142506$

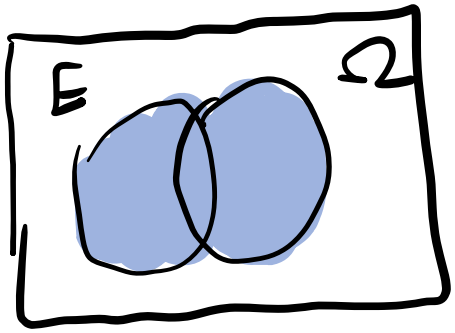
Now you know how to count.

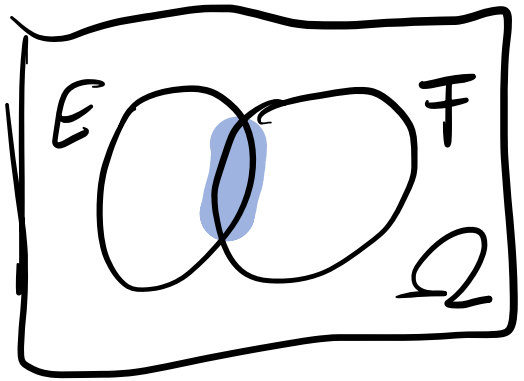
Events

Events

A mathematical model for experiments:

- *Sample space*: the set Ω of all possible outcomes
- An *event* is a collection[1] of possible outcomes: $E \subseteq \Omega$
- *Union* $E \cup F$ and *intersection* $E \cap F$ of events make sense





[1] Sometimes Ω is too large, and not all subsets are events. Ignore this now.

Events

A mathematical model for experiments:

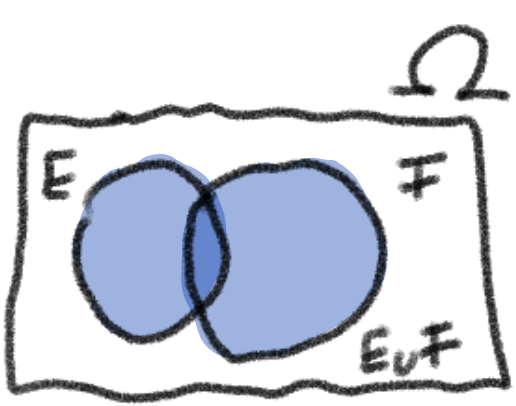
- *Sample space*: the set Ω of all possible outcomes
- An *event* is a collection[1] of possible outcomes: $E \subseteq \Omega$
- *Union* $E \cup F$ and *intersection* $E \cap F$ of events make sense

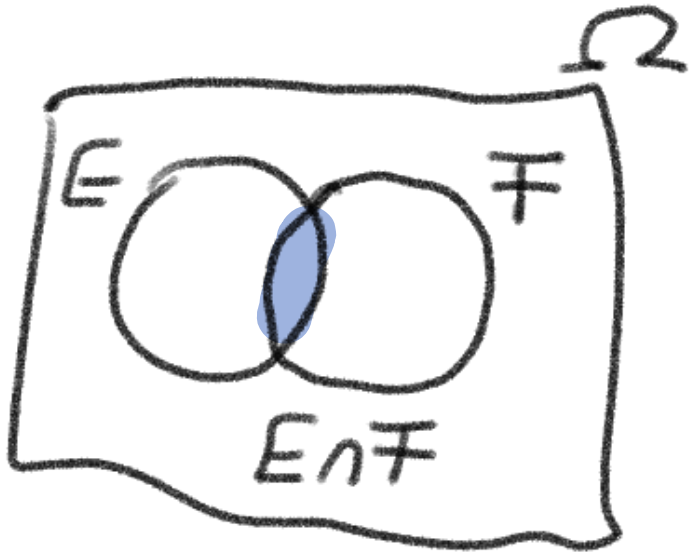
[1] Sometimes Ω is too large, and not all subsets are events. Ignore this now.

Events

A mathematical model for experiments:

- *Sample space*: the set Ω of all possible outcomes
- An *event* is a collection[1] of possible outcomes: $E \subseteq \Omega$
- *Union* $E \cup F$ and *intersection* $E \cap F$ of events make sense





[1] Sometimes Ω is too large, and not all subsets are events. Ignore this now.

Examples

Experiment	Sample Space	Event
Will it rain	$\Omega = \{r, n\}$	e.g $E = \{r\}$
5-horse race	Ω = permutations of $\{A,B,C,D,E\}$ $\lvert\Omega\rvert = 5!$	e.g $E = \{$ B wins $\}$ $\lvert E\rvert = 4!$ $F = \{$ B wins, A third $\}$ $\lvert F\rvert = 3!$
flip 2 coins	$\Omega = \{(H,H)(H,T)$ $(T,H)(T,T)\}$	E = different coins $\{(H,T),(T,H)\}$ F = at least 1 head $\{(H,T)(T,H)(H,H)\}$
roll a dice until we get 6	$\Omega = \{$ sequences 1-5 followed by 6	E = first 4 then 6 at 3rd roll $\{(4,1,6),(4,2,6) \cdots (4,5,6)\}$

Examples

Experiment	Sample Space	Event
Will it rain?	$\Omega = \{r, n\}$	e.g. $E = \{r\}$

Examples

Experiment	Sample Space	Event
Will it rain?	$\Omega = \{r, n\}$	e.g. $E = \{r\}$
5-horse race	$\Omega =$ permutations of $\{A, B, C, D, E\}$	e.g. $E = \{B \text{ wins}\}$ $F = \{E \text{ wins, A third}\}$

Examples

Experiment	Sample Space	Event
Will it rain?	$\Omega = \{r, n\}$	e.g. $E = \{r\}$
5-horse race	$\Omega =$ permutations of $\{A, B, L, D, E\}$	e.g. $E = \{B \text{ wins}\}$ $F = \{E \text{ wins, A third}\}$
flip 2 coins	$\Omega = \{(H,H), (H,T), (T,H), (H,H)\}$	e.g. $E =$ coins different $= \{(H,T)\ (T,H)\}$

Examples

Experiment	Sample Space	Event
Will it rain?	$\Omega = \{r, n\}$	e.g. $E = \{r\}$
5-horse race	Ω = permutations of $\{A, B, L, D, E\}$	e.g. $E = \{B \text{ wins}\}$ $F = \{E \text{ wins, A third}\}$
flip 2 coins	$\Omega = \{(H,H), (H,T), (T,H), (H,H)\}$	e.g. E = coins different $= \{(H,T), (T,H)\}$ F = at least one head $= \{(H,T), (T,H), (H,H)\}$
roll dice until 1 get 6	$\Omega = \{$sequences with 1 before 6$\}$	E = first 4, then 6 on 3rd roll $= \{(4,1,6), (4,2,6), (4,3,6) \ldots\}$

Union and intersection

Union

Union $E \cup F$ of events E and F means E or F.
Infinite union $\bigcup_i E_i$ of events E_i means at least one of the E_i's.

Intersection

Intersection $E \cap F$ of events E and F means E and F.
Infinite intersection $\bigcap_i E_i$ of events E_i means each of the E_i's.

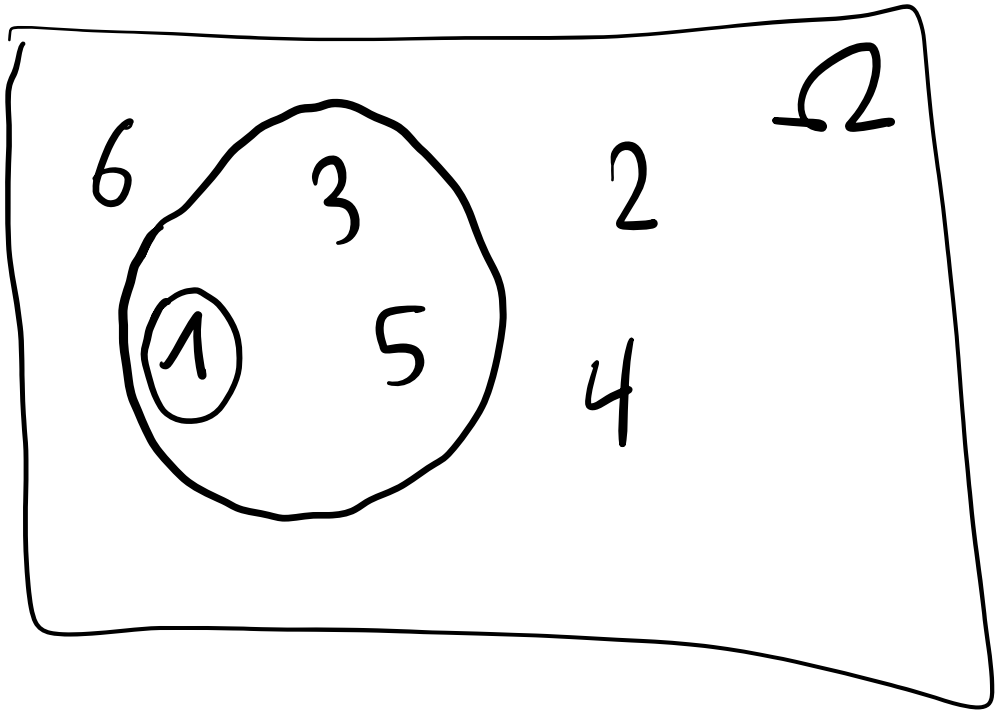
Definition

If $E \cap F = \emptyset$, we call events E and F *mutually exclusive*.
If events $E_1, E_2, \ldots$ satisfy $E_i \cap E_j = \emptyset$ whenever $i \neq j$, we call them *mutually exclusive*. They cannot happen at the same time.

Inclusion and implication

Remark

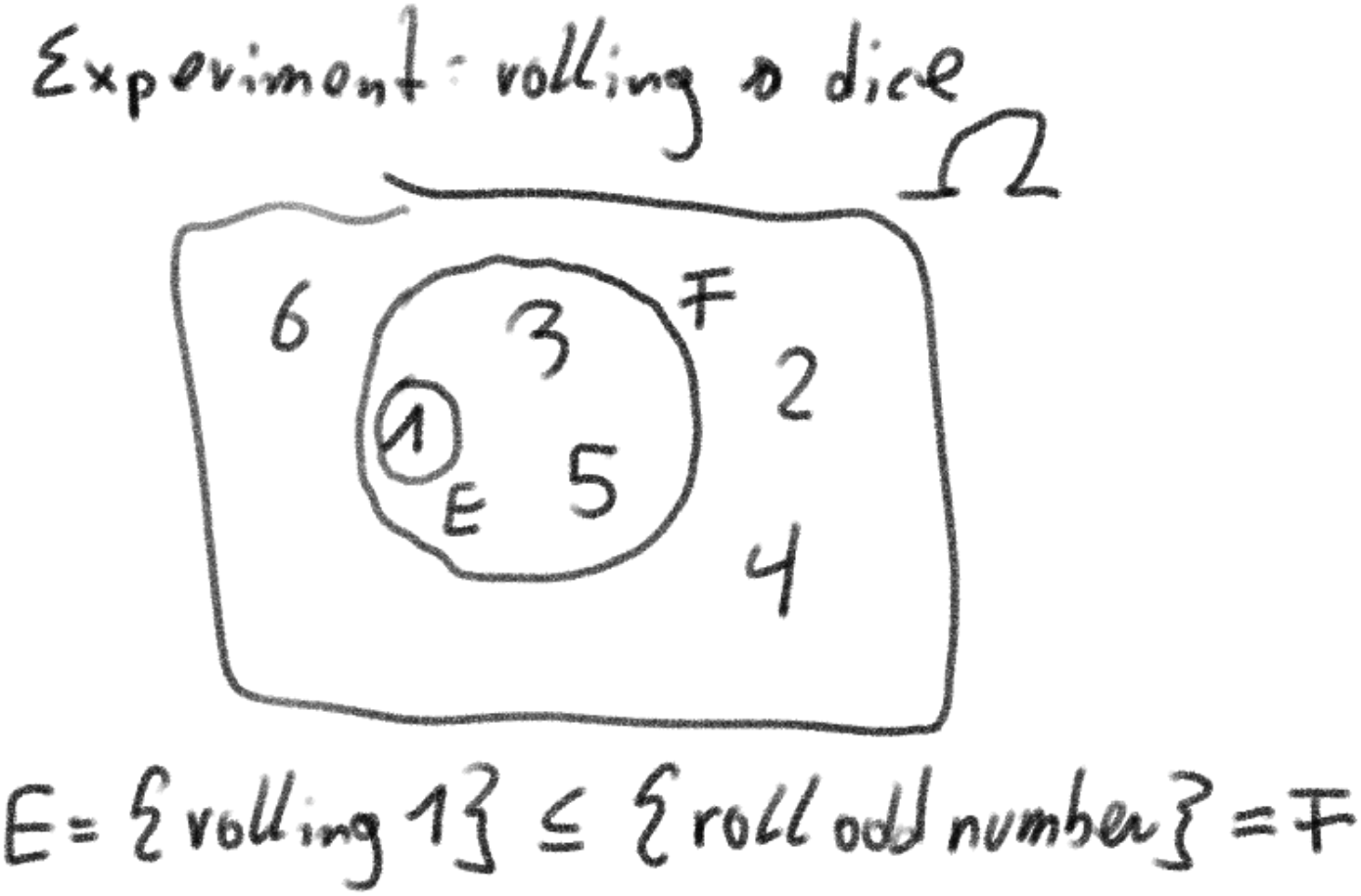
If the event E is a *subset* of the event F, written $E \subseteq F$, then the occurrence of E *implies* that of F.



Inclusion and implication

Remark

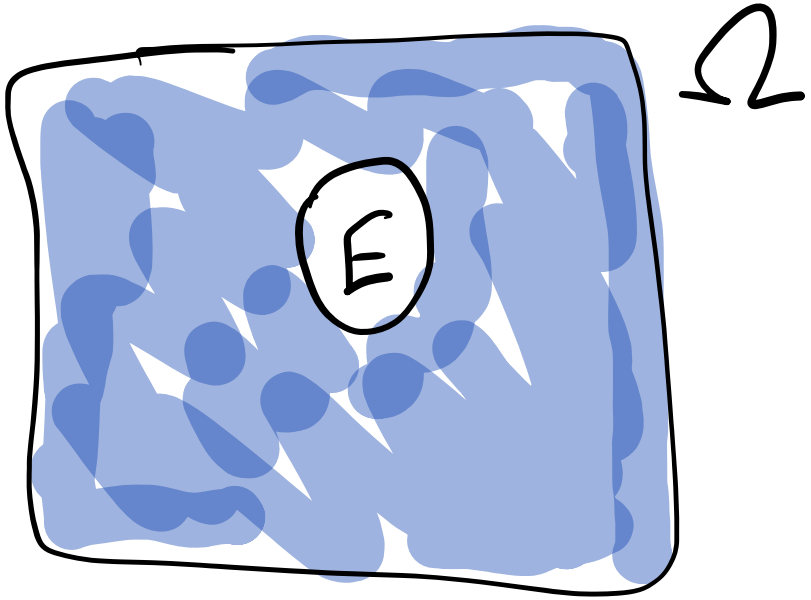
If the event E is a *subset* of the event F, written $E \subseteq F$, then the occurrence of E *implies* that of F.



Complementarity

Definition

The *complement* of an event E is $E^c = \Omega - E$.
This is the event that E does *not* occur.



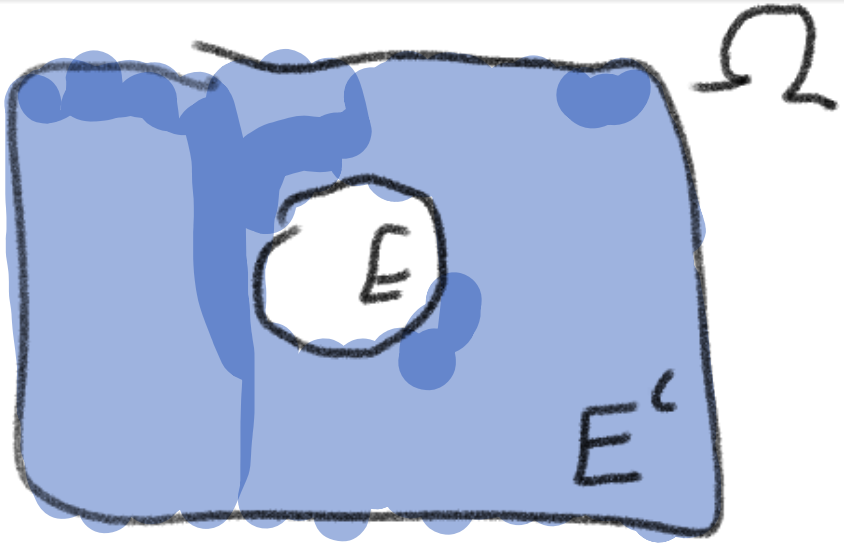
$E \cap E' = \emptyset \quad E \cup E' = \Omega$

Complementarity

Definition

The *complement* of an event E is $E^c = \Omega - E$.

This is the event that E does *not* occur.



$E \cap E^c = \emptyset \qquad E \cup E^c = \Omega$

Properties of events

- Commutativity: $E \cup F = F \cup E$
  $E \cap F = F \cap E$

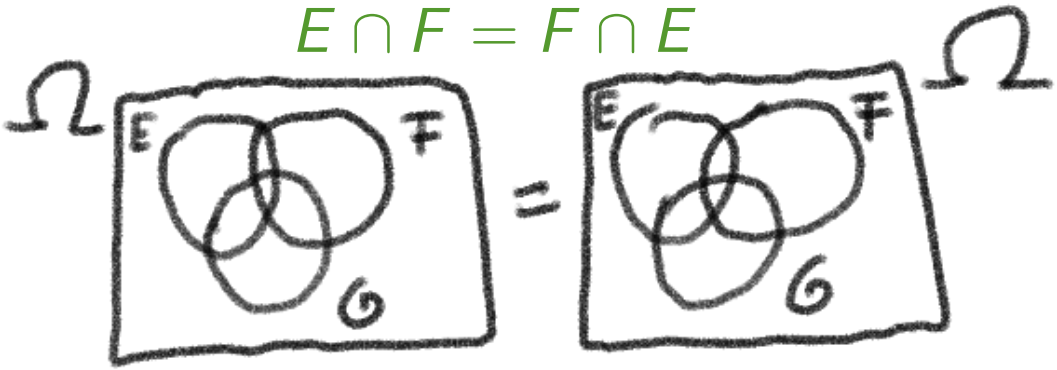
- Associativity: $E \cup (F \cup G) = (E \cup F) \cup G$
  $E \cap (F \cap G) = (E \cap F) \cap G$

Properties of events



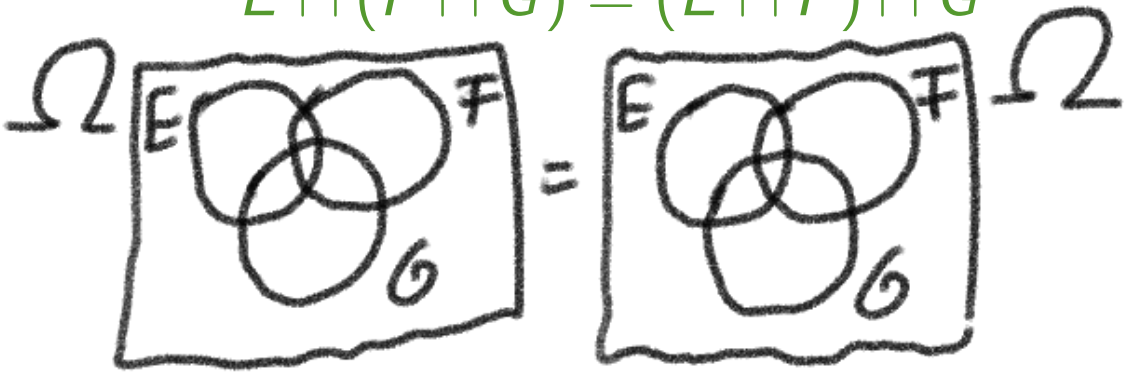
- Commutativity: $E \cup F = F \cup E$

  $E \cap F = F \cap E$



- Associativity: $E \cup (F \cup G) = (E \cup F) \cup G$

  $E \cap (F \cap G) = (E \cap F) \cap G$

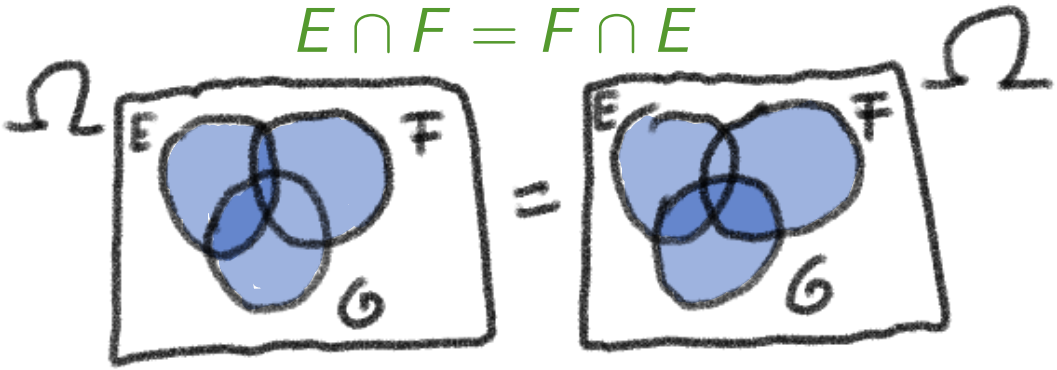


Properties of events



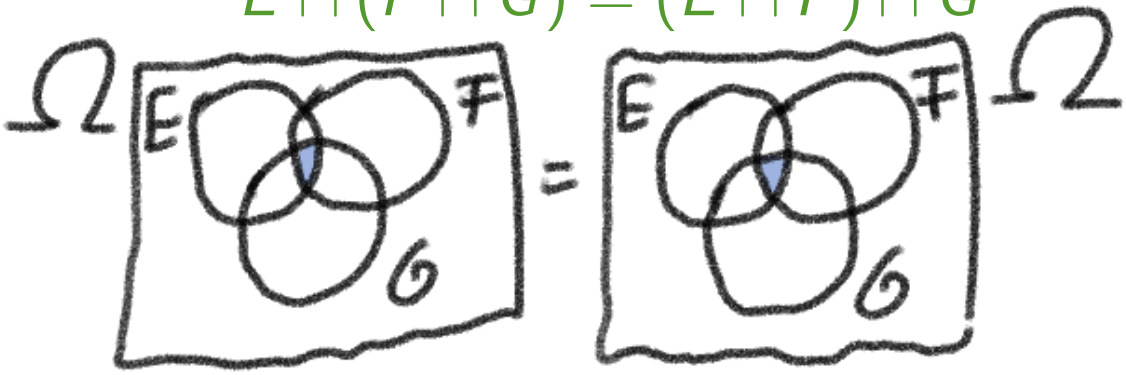
- Commutativity: $E \cup F = F \cup E$

  $E \cap F = F \cap E$



- Associativity: $E \cup (F \cup G) = (E \cup F) \cup G$

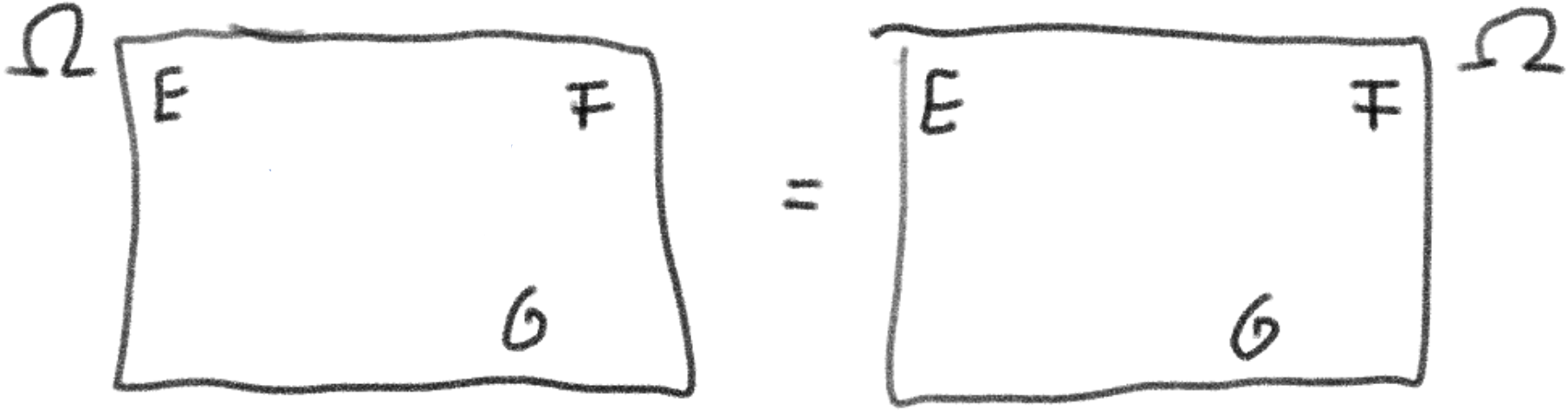
  $E \cap (F \cap G) = (E \cap F) \cap G$



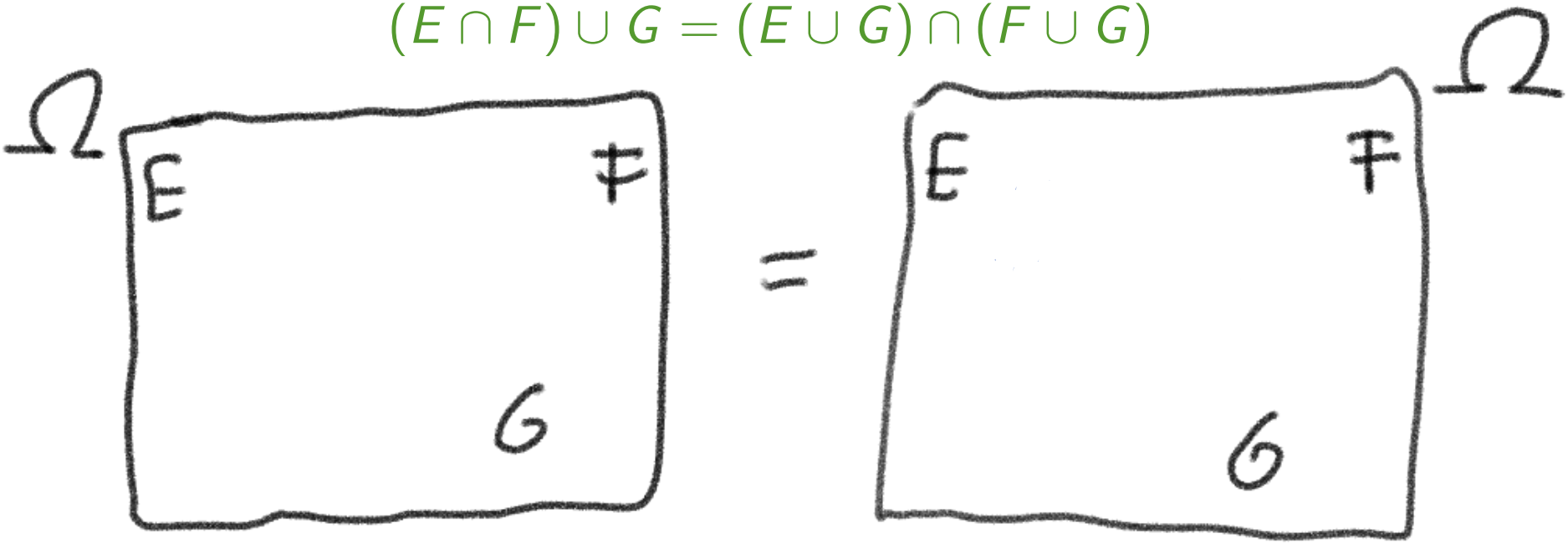
Properties of events

- Distributivity: $(E \cup F) \cap G = (E \cap G) \cup (F \cap G)$
  $(E \cap F) \cup G = (E \cup G) \cap (F \cup G)$

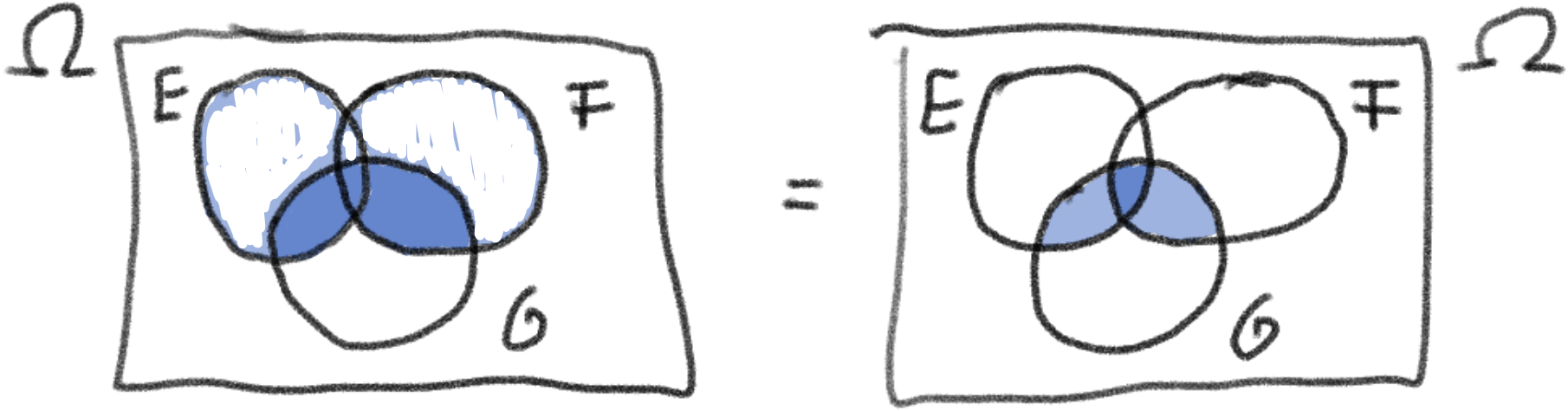
Properties of events



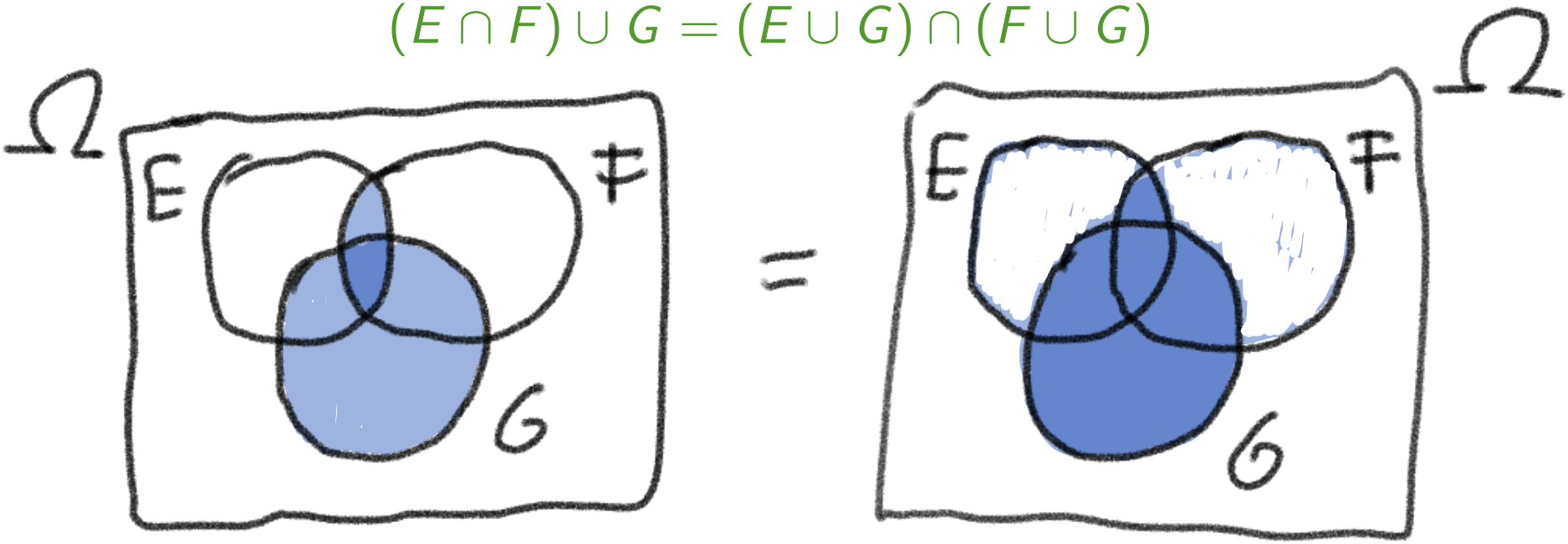
- Distributivity: $(E \cup F) \cap G = (E \cap G) \cup (F \cap G)$
  $(E \cap F) \cup G = (E \cup G) \cap (F \cup G)$



Properties of events



- Distributivity: $(E \cup F) \cap G = (E \cap G) \cup (F \cap G)$
  $(E \cap F) \cup G = (E \cup G) \cap (F \cup G)$



De Morgan's law

- De Morgan's law: $(E \cup F)^c = E^c \cap F^c$

De Morgan's law

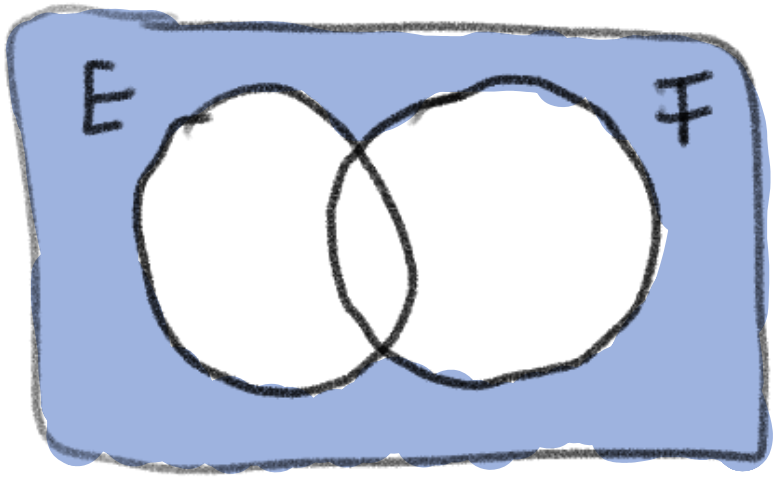
E = you have umbrella
F = I have umbrella

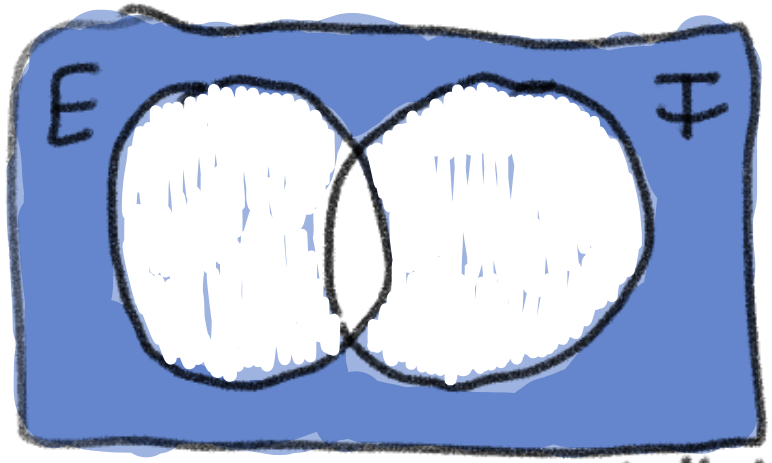
- De Morgan's law: $(E \cup F)^c = E^c \cap F^c$ = not true that one of us has umbrella = both have no umbrella

De Morgan's law

E = you have umbrella
F = I have umbrella

- De Morgan's law: $(E \cup F)^c = E^c \cap F^c$ = not true that one of us has umbrella





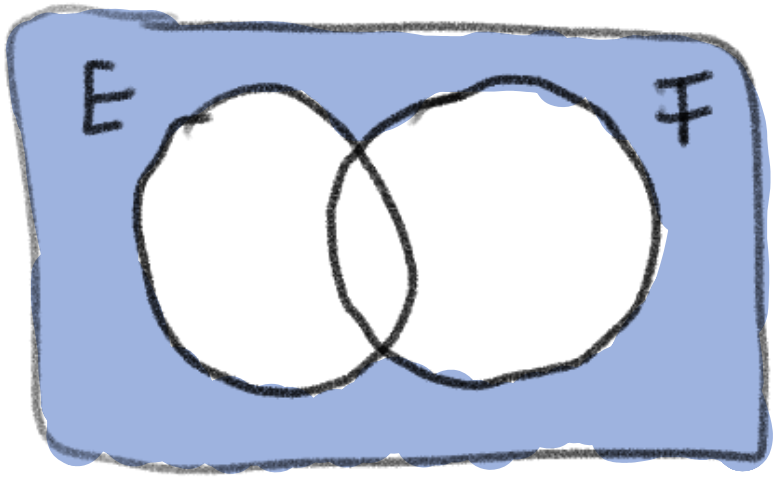
= both have no umbrella

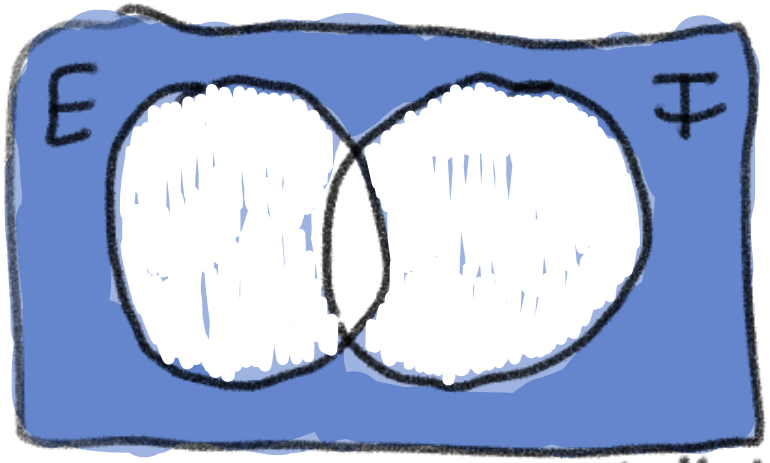
$(E \cap F)^c = E^c \cup F^c$ = not true both have umbrella
= one does not have umbrella

De Morgan's law

E = you have umbrella
F = I have umbrella

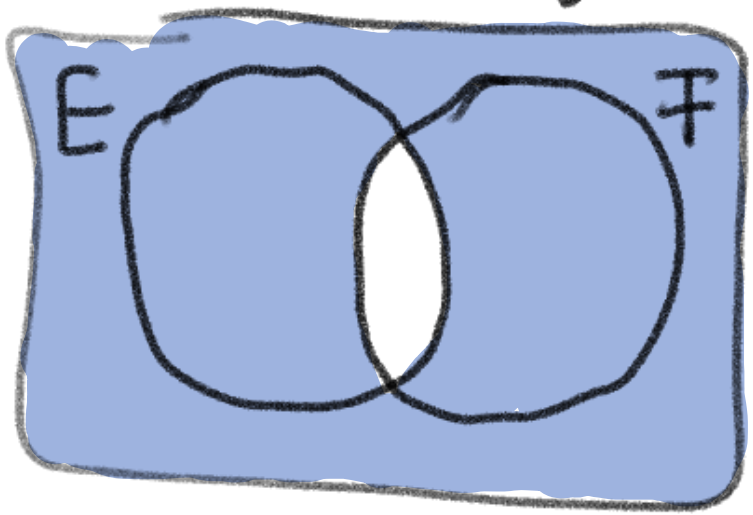
- De Morgan's law: $(E \cup F)^c = E^c \cap F^c$ = not true that one of us has umbrella

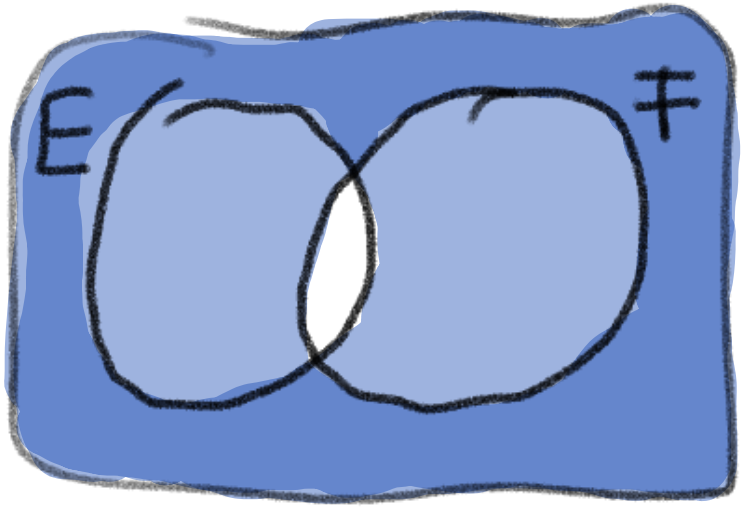




= both have no umbrella

$(E \cap F)^c = E^c \cup F^c$ = not true both have umbrella = one does not have umbrella





Probability

So far been structural
now we will attach numbers to things

- Definition by axioms
- How to compute probabilities
- Inclusion-exclusion principle
- Equally likely outcomes

Axioms of probability

Definition

The *probability* $\mathbf{P}$ on a sample space Ω assigns *numbers* to *events* of Ω in such a way that:

1. the probability of any event is non-negative: $\mathbf{P}(E) \geq 0$;
2. the probability of the sample space is one: $\mathbf{P}(\Omega) = 1$;
3. for countably many *mutually exclusive* events $E_1, E_2, \ldots$:

$$\mathbf{P}\left(\bigcup_i E_i\right) = \sum_i \mathbf{P}(E_i)$$

Axioms of probability

Definition

P is a function that satisfies 3 axioms

The *probability* **P** on a sample space Ω assigns *numbers* to *events* of Ω in such a way that:

1. the probability of any event is non-negative: $\mathbf{P}(E) \geq 0$;
2. the probability of the sample space is one: $\mathbf{P}(\Omega) = 1$;
3. for countably many *mutually exclusive* events E_1, E_2, ...:

$$\mathbf{P}(\bigcup_i E_i) = \sum_i \mathbf{P}(E_i)$$

finite $P(E_1 \cup \cdots \cup E_n) = P(E_1) + \cdots + P(E_n)$

countably infinite $P(E_1 \cup E_2 \cup E_3 \cup \ldots) = P(E_1) + P(E_2) + P(E_3) + \cdots$

How to compute probabilities

Proposition $P(E) = 1 - P(E^c)$

For any event, $\mathbf{P}(E^c) = 1 - \mathbf{P}(E)$.

Proof E, E^c $E \cap E^c = \emptyset$ Axiom 3

$P(E) + P(E^c) = P(\Omega)$ Axiom 2

$= 1$

Corollary

We have $\mathbf{P}(\emptyset) = \mathbf{P}(\Omega^c) = 1 - \mathbf{P}(\Omega) = 1 - 1 = 0$.

For any event, $\mathbf{P}(E) = 1 - \mathbf{P}(E^c) \leq 1$.

How to compute probabilities

Proposition

For any event, $\mathbf{P}(E^c) = 1 - \mathbf{P}(E)$.

Proof: E and E^c are mutually exclusive $E \cap E^c = \emptyset$

so by axiom 3 $P(E) + P(E^c) = P(\Omega)$

by axiom 2 $= 1$

Corollary

1) *We have* $\mathbf{P}(\emptyset) = \mathbf{P}(\Omega^c) = 1 - \mathbf{P}(\Omega) = 1 - 1 = 0$.
2) *For any event,* $\mathbf{P}(E) = 1 - \mathbf{P}(E^c) \leq 1$.

by axiom 1 ≥ 0

How to compute probabilities

Proposition

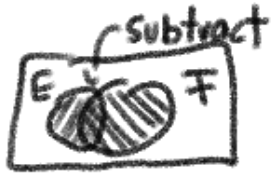
For any two events, $\mathbf{P}(E \cup F) = \mathbf{P}(E) + \mathbf{P}(F) - \mathbf{P}(E \cap F)$.

Proposition (*Boole's inequality*)

For any events $E_1, E_2, \ldots, E_n$*:*

$$\mathbf{P}\left(\bigcup_{i=1}^{n} E_i\right) \leq \sum_{i=1}^{n} \mathbf{P}(E_i).$$

How to compute probabilities



Proposition

For any two events, $\mathbf{P}(E \cup F) = \mathbf{P}(E) + \mathbf{P}(F) - \mathbf{P}(E \cap F)$.

Proposition (*Boole's inequality*)

For any events $E_1, E_2, \ldots, E_n$*:*

$$\mathbf{P}\left(\bigcup_{i=1}^{n} E_i\right) \leq \sum_{i=1}^{n} \mathbf{P}(E_i).$$

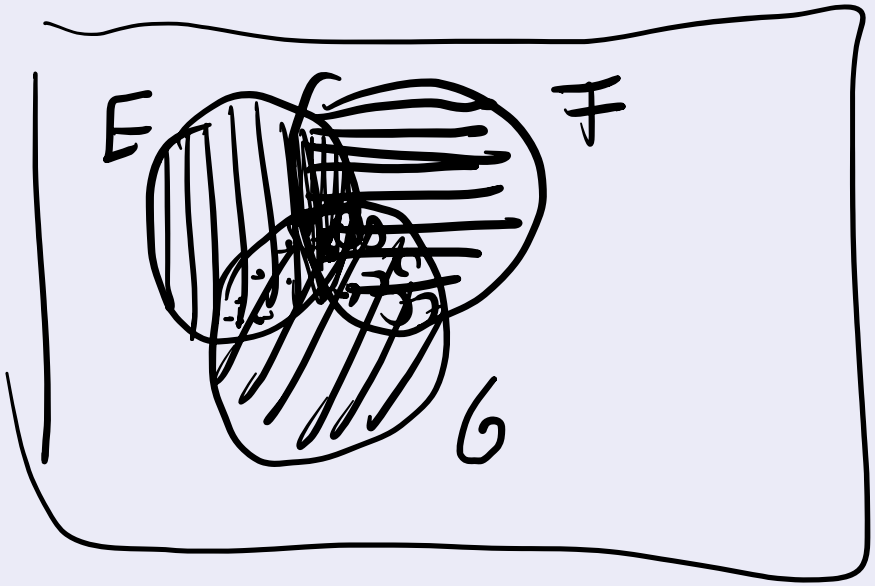
skipping proof by induction initially

Inclusion-exclusion

Proposition

For any events:

$$\begin{aligned}\mathbf{P}(E \cup F \cup G) = \mathbf{P}(E) + \mathbf{P}(F) + \mathbf{P}(G) \\ - \mathbf{P}(E \cap F) - \mathbf{P}(E \cap G) - \mathbf{P}(F \cap G) \\ + \mathbf{P}(E \cap F \cap G).\end{aligned}$$

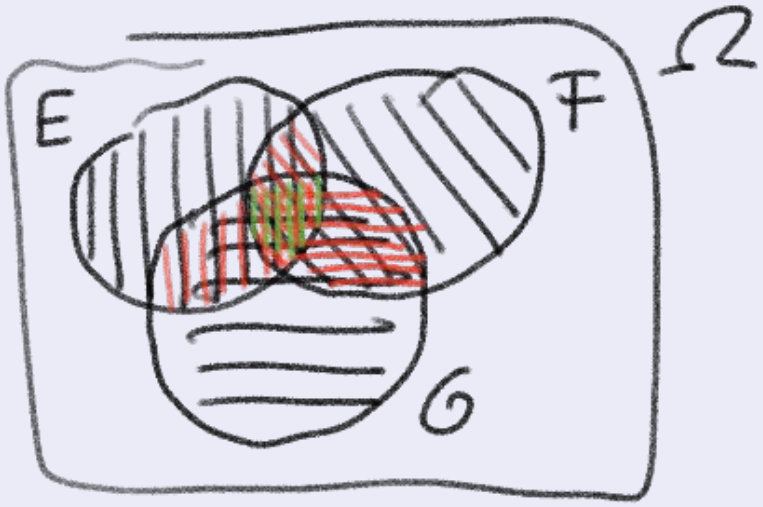


Inclusion-exclusion

Proposition

For any events:

$$\begin{aligned}\mathbf{P}(E \cup F \cup G) = \mathbf{P}(E) + \mathbf{P}(F) + \mathbf{P}(G) \\ - \mathbf{P}(E \cap F) - \mathbf{P}(E \cap G) - \mathbf{P}(F \cap G) \\ + \mathbf{P}(E \cap F \cap G).\end{aligned}$$



Inclusion-exclusion

Proposition

For any events:

$$\begin{aligned}\mathbf{P}(E \cup F \cup G) = \mathbf{P}(E) + \mathbf{P}(F) + \mathbf{P}(G) \\ - \mathbf{P}(E \cap F) - \mathbf{P}(E \cap G) - \mathbf{P}(F \cap G) \\ + \mathbf{P}(E \cap F \cap G).\end{aligned}$$

$$\begin{aligned}\mathbf{P}(E_1 \cup E_2 \cup \cdots \cup E_n) = \sum_{1 \le i \le n} \mathbf{P}(E_i) \\ - \sum_{1 \le i_1 < i_2 \le n} \mathbf{P}(E_{i_1} \cap E_{i_2}) \\ + \sum_{1 \le i_1 < i_2 < i_3 \le n} \mathbf{P}(E_{i_1} \cap E_{i_2} \cap E_{i_3}) \\ - \cdots \\ + (-1)^{n+1} \mathbf{P}(E_1 \cap E_2 \cap \cdots \cap E_n).\end{aligned}$$

all the way up to n

Example

Example

In a sports club, with N members

36 members play tennis, 22 play tennis and squash,
28 play squash, 12 play tennis and badminton,
18 play badminton, 9 play squash and badminton,
4 play tennis, squash and badminton.

What is probability that a random member
~~How many~~ plays at least one of these games?

Example

Example

In a sports club, with N members

36 members play tennis, 22 play tennis and squash,
28 play squash, 12 play tennis and badminton,
18 play badminton, 9 play squash and badminton,
4 play tennis, squash and badminton.

What is probability that a random member
~~How many~~ play at least one of these games?

$$P(T \cup S \cup B) = P(T) + P(S) + P(B) - P(T \cap S) - P(T \cap B) - P(S \cap B) + P(T \cap S \cap B)$$

$$= \frac{36}{N} + \frac{28}{N} + \frac{18}{N} - \frac{22}{N} - \frac{12}{N} - \frac{9}{N} + \frac{4}{N} = \frac{43}{N}$$

How to compute probabilities

Proposition

If $E \subseteq F$, *then* $\mathbf{P}(F - E) = \mathbf{P}(F) - \mathbf{P}(E)$.

Corollary

If $E \subseteq F$, *then* $\mathbf{P}(E) \leq \mathbf{P}(F)$.

Equally likely outcomes

The return of counting

The last thing before we all go home I want to talk about this

Finite sample space, $|\Omega| = N < \infty$, has special important case where each experiment outcome has *equal probability*:

$$\mathbf{P}(\omega) = \frac{1}{N} \qquad \text{for all } \omega \in \Omega$$

Definition

Outcomes $\omega \in \Omega$ are also called *elementary events*.

Example

Example

Rolling two dice, what is the probability that the sum of the numbers shown is 7?

What's wrong with this solution? "The number 7 is one out of the possible values $2, 3, \ldots, 12$ for the sum, and the answer is $\frac{1}{11}$."

what is wrong with the answer?

Example

Example

Rolling two dice, what is the probability that the sum of the numbers shown is 7?

What's wrong with this solution? "The number 7 is one out of the possible values $2, 3, \ldots, 12$ for the sum, and the answer is $\frac{1}{11}$."

sums are not equally likely:
e.g. 12 is only 1 out of 36.

what is wrong with the answer?

Example

Example

Rolling two dice, what is the probability that the sum of the numbers shown is 7?

What's wrong with this solution? "The number 7 is one out of the possible values $2, 3, \ldots, 12$ for the sum, and the answer is $\frac{1}{11}$."

sums are not equally likely: e.g. 12 is only 1 out of 36.

what is wrong with the answer?

$$\Omega = \{(i,j) \mid i,j \in \{1,2,\ldots,6\}\}$$

$$E = \text{sum is } 7 = \{(i,j) \mid i+j=7\} = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$$

$$P(E) = \frac{6}{36} = \frac{1}{6}$$

Summary

- Counting: permutations, combinations, repetitions
- Events: sample space, union, intersection, complement
- Experiments: distributivity, De Morgan's law
- Probability: axioms, how to compute, equally likely outcomes