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Task A

You plan to visit some tourist attractions in Edinburgh, going to a different one every week. There are 14 different attractions that you are interested in but sadly you only have six weeks available.

(a) How many choices do you have for which six attractions to visit?

(b) Another way to approach this is to think of what your six-week plan will be, visiting each week a different attraction out of the 14. How many possible plans are there?

(c) How do your answers to the first two parts relate to each other?

(d) Suppose the numbers were the other way around: you had 14 weeks and only six attractions, so some weeks you will not go to see anything. How many different plans are possible?

You can leave factorials in your answers unevaluated, like 7! or 12!, but not binomial coefficients like $\binom{4}{2}$ or ${}_6\mathrm{C}_2$.

Task B

The *Scottish Weather Research Institute* has sixteen members and is made up of two distinct research groups: seven researchers study wind, while the other nine study rain. The institute is run by a committee of five people chosen from its members.

(a) How many ways are there to select a committee of five institute members if at least one wind researcher must be on the committee?

(b) How many ways are there to select a committee of five institute members if both wind and rain groups must have at least one member on the committee?

Task C

A telegraph sends out three symbols in succession on its communication line. Some symbols are received successfully, others are missed.

(a) Use 1 for a symbol successfully received and 0 for a missed symbol. List all possible outcomes as strings of 3 bits.

(b) Converting binary strings to decimal, list all outcomes as integers.

(c) Taking these integers as a sample space, represent the following events as sets on a single Venn diagram.

$$A_1 = \{\text{only the first symbol is received}\}$$
$$A_2 = \{\text{at least one symbol is received}\}$$
$$A_3 = \{\text{exactly two symbols are received}\}$$
$$A_4 = \{\text{less than two symbols are received}\}$$
$$A_5 = \{\text{exactly one symbol is received}\}$$

Task D

Five cards are numbered 1, 2, 3, 4, and 5. Three cards are randomly selected from these five and are lined up next to each other to form a 3-digit number x. Find the probabilities of the following events.

(a) $A = \{x = 123\}$

(b) $B = \{x$ does not contain the digit $4\}$

(c) $C = \{x$ is even$\}$

(d) $D = \{x$ contains at least one of the digits 1, 2$\}$

Task E

(a) How many functions $f{:}\{1, 2, 3, 4, 5\} \to \{a, b, c, d\}$ are there?

(b) How many functions $f{:}\{1, 2, 3, 4, 5, 6, 7, 8\} \to \{1, 2, 3, 4, 5, 6, 7, 8\}$ are bijective?

(c) How many functions $f{:}\{1, 2, 3\} \to \{1, 2, 3, 4, 5, 6, 7, 8\}$ are injective?

Task F

In how many ways can you order the elements of the set $\{1, 2, \ldots, 2n\}$ so that every even number is at an even position?

Solution Notes

Task A

(a) You need to choose 6 attractions from a group of 14 so there are $\binom{14}{6} = \frac{14!}{8!6!} = 3003$ ways.

(b) Here you must count all the ordered ways to choose 6 attractions from 14. This means all the permutations ${}_{14}\mathrm{P}_6 = \frac{14!}{(14-6)!} = \frac{14!}{8!} = 2162160$.

(c) In both counting problems you choose 6 out of 14 attractions. For the first one, that's it, with 3003 ways. But for the second counting problem, each of those 3003 choices of 6 attractions can be arranged in 6! ways. So now there are $6! \times 3003 = 2162160$ choices and that is exactly ${}_{14}\mathrm{P}_6$.

(d) Now for each of 6 attractions you need to pick which week you will visit them: for the first you have 14 possibilities, for the second 13, and so on. This makes a total of $\frac{14!}{(14-6)!} = {}_{14}\mathrm{P}_6 = 2162160$ possible choices.

Task B

(a) The total number of ways to select a committee with 5 members of the institute is $\binom{16}{5}$. Of those, $\binom{9}{5}$ ways have only rain researchers. Thus, those with at least one wind researcher are $\binom{16}{5} - \binom{9}{5} = 4242$.

Alternatively, we can divide into four cases depending on how many wind researchers there are count each separately: $\binom{7}{1}\binom{9}{4}$ committees have exactly one wind researcher, $\binom{7}{2}\binom{9}{3}$ have exactly two, $\binom{7}{3}\binom{9}{2}$ have three, $\binom{7}{4}\binom{9}{1}$ have four, and $\binom{7}{5}$ have all five members studying wind. From this we calculated as follows.

$$\begin{aligned}\binom{7}{1}\binom{9}{4} + \binom{7}{2}\binom{9}{3} + \binom{7}{3}\binom{9}{2} + \binom{7}{4}\binom{9}{1} + \binom{7}{5} \\ = \; 7 \cdot 126 + 21 \cdot 84 + 35 \cdot 36 + 35 \cdot 9 + 21 \;=\; 4242\end{aligned}$$

(b) From a total of $\binom{16}{5}$ possible committees we want to eliminate the $\binom{9}{5}$ choices where all members study rain and the $\binom{7}{5}$ where they are all wind researchers. The calculation is then as follows.

$$\binom{16}{5} - \binom{9}{5} - \binom{7}{5} \;=\; 4368 - 126 - 21 \;=\; 4221$$

Again, we can also calculate it by counting up the number of committees with one wind researcher and four rain researchers, and so on.

$$\begin{aligned}\binom{7}{1}\binom{9}{4} + \binom{7}{2}\binom{9}{3} + \binom{7}{3}\binom{9}{2} + \binom{7}{4}\binom{9}{1} &= 7 \cdot 126 + 21 \cdot 84 + 35 \cdot 36 + 35 \cdot 9 \\ &= 4221.\end{aligned}$$

Task C

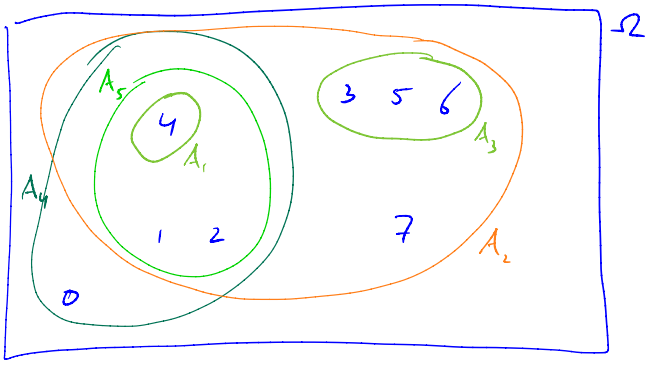
This table shows the possible outcomes as three-bit strings and as integers.

000	001	010	011	100	101	110	111
0	1	2	3	4	5	6	7

Take the sample space as $\mathcal{S} = \{0, 1, 2, 3, 4, 5, 6, 7\}$, the events are as follows.

$$
\begin{aligned}
A_1 &= \{4\} \\
A_2 &= \{1, 2, 3, 4, 5, 6, 7\} \\
A_3 &= \{3, 5, 6\} \\
A_4 &= \{0, 1, 2, 4\} \\
A_5 &= \{1, 2, 4\}
\end{aligned}
$$

The following Venn diagram presents all of these together as overlapping sets.



Task D

The number of possible permutations of 3 elements from a set of 5 is ${}_5P_3 = 5 \cdot 4 \cdot 3 = 60$.

(a) Only the permutation 123 is selected, so the event A has cardinality 1, and thus probability $P(A) = \frac{1}{60}$.

(b) To not contain the digit 4, we can choose only among digits 1, 2, 3, 5, for which there are ${}_4P_3 = 4 \cdot 3 \cdot 2 = 24$ possibilities, so $P(B) = \frac{24}{60} = \frac{2}{5}$.

(c) For x to be even, its last digit must be 2 or 4. In each of those cases, the first two digits can have one of 4 values (1,3,4,5 in the first case, and 1,2,3,5 in the second case). Therefore event C has cardinality $2 \cdot {}_4P_2 = 2 \cdot 4 \cdot 3 = 24$, and $P(C) = \frac{24}{60} = \frac{2}{5}$.

(d) Consider the complement $D^c = \{x \text{ does not contain the digits 1 or 2}\}$. Then every digit of x can be one of three values 3, 4, or 5, and so D^c has cardinality ${}_3P_3 = 3! = 6$, and so $P(D^c) = \frac{6}{60} = \frac{1}{10}$, and therefore $P(D) = 1 - P(D^c) = 1 - \frac{1}{10} = \frac{9}{10}$.

Task E

(a) $4^5 = 1024$

(b) $8! = 40320$

(c) ${}_8\mathrm{P}_3 = 336$

Task F

There are n even numbers in the set. There are also n even positions in a sequence of $2n$ numbers. The number of ways to order n numbers in n positions is $n!$. Clearly, there are also n odd numbers in the set and they can also be ordered in n odd positions in $n!$ possible ways. So for each ordering of even numbers we have $n!$ orderings of odd numbers. Therefore the total number of orderings is $n! \cdot n! = (n!)^2$.