

# Elements of Programming Languages

## Lecture Notes: $L_{\text{Data}}$

### 1 Abstract Syntax

|                 |   |  |
|-----------------|---|--|
| $Expr \ni e$    | $::=$ $n \in \mathbb{N} \mid e_1 + e_2 \mid e_1 \times e_2$<br>$\mid b \in \mathbb{B} \mid e_1 == e_2 \mid \text{if } e \text{ then } e_1 \text{ else } e_2$<br>$\mid x \mid \text{let } x = e_1 \text{ in } e_2$<br>$\mid e_1 e_2 \mid \lambda x:\tau. e$<br>$\mid \text{rec } f(x : \tau_1) : \tau_2. e$<br>$\mid (e_1, e_2) \mid \text{fst } e \mid \text{snd } e \mid \text{let pair } (x, y) = e_1 \text{ in } e_2 \mid ()$<br>$\mid \text{left}(e) \mid \text{right}(e) \mid \text{case } e \text{ of } \{\text{left}(x) \Rightarrow e_1 ; \text{right}(y) \Rightarrow e_2\}$ | $L_{\text{Arith}}$<br>$L_{\text{If}}$<br>$L_{\text{Let}}$<br>$L_{\text{Lam}}$<br>$L_{\text{Rec}}$<br>$L_{\text{Data}}$ |
| $Type \ni \tau$ | $::=$ $\text{int}$<br>$\mid \text{bool}$<br>$\mid \tau_1 \rightarrow \tau_2$<br>$\mid \tau_1 \times \tau_2 \mid \text{unit} \mid \tau_1 + \tau_2 \mid \text{empty}$   | $L_{\text{Arith}}$<br>$L_{\text{If}}$<br>$L_{\text{Lam}}$<br>$L_{\text{Data}}$   |
| $\Gamma$        | $::=$ $x_1 : \tau_1, \dots, x_n : \tau_n$   |  |
| $Value \ni v$   | $::=$ $n \in \mathbb{N}$<br>$\mid b \in \mathbb{B}$<br>$\mid \lambda x. e$<br>$\mid \text{rec } f(x). e$<br>$\mid (v_1, v_2) \mid () \mid \text{left}(v) \mid \text{right}(v)$  | $L_{\text{Arith}}$<br>$L_{\text{If}}$<br>$L_{\text{Lam}}$<br>$L_{\text{Rec}}$<br>$L_{\text{Data}}$                     |

## 1.1 Free variables

In the following,  $\oplus$  stands for any binary operator.

$$\begin{aligned}
FV(n) &= \emptyset \\
FV(e_1 \oplus e_2) &= FV(e_1) \cup FV(e_2) \\
FV(b) &= \emptyset \\
FV(\text{if } e \text{ then } e_1 \text{ else } e_2) &= FV(e) \cup FV(e_1) \cup FV(e_2) \\
FV(x) &= \{x\} \\
FV(\text{let } x = e_1 \text{ in } e_2) &= FV(e_1) \cup (FV(e_2) - \{x\}) \\
FV(e_1 e_2) &= FV(e_1) \cup FV(e_2) \\
FV(\lambda x:\tau. e) &= FV(e) - \{x\} \\
FV(\text{rec } f(x:\tau) : \tau'. e) &= FV(e) - \{f, x\} \\
FV((e_1, e_2)) &= FV(e_1) \cup FV(e_2) \\
FV(\text{fst } e) &= FV(e) \\
FV(\text{snd } e) &= FV(e) \\
FV(\text{let } (x, y) = e_1 \text{ in } e_2) &= FV(e_1) \cup (FV(e_2) - \{x, y\}) \\
FV(\text{left}(e)) &= FV(e) \\
FV(\text{right}(e)) &= FV(e) \\
FV(\text{case } e \text{ of } \{\text{left}(x) \Rightarrow e_1 ; \text{right}(y) \Rightarrow e_2\}) &= FV(e) \cup (FV(e_1) - \{x\}) \cup (FV(e_2) - \{y\})
\end{aligned}$$

## 1.2 Substitution

$$\begin{aligned}
n[e/x] &= n \\
(e_1 \oplus e_2)[e/x] &= e_1[e/x] \oplus e_2[e/x] \\
b[e/x] &= b \\
(\text{if } e_0 \text{ then } e_1 \text{ else } e_2)[e/x] &= \text{if } (e_0[e/x]) \text{ then } (e_1[e/x]) \text{ else } (e_2[e/x]) \\
x[e/x] &= e \\
y[e/x] &= y \quad (x \neq y) \\
(\text{let } y = e_1 \text{ in } e_2)[e/x] &= \text{let } y = e_1[e/x] \text{ in } e_2[e/x] \\
&\quad (\text{where } y \# e) \\
(\lambda y:\tau. e_0)[e/x] &= \lambda y:\tau. e_0[e/x] \\
&\quad (\text{where } y \# e) \\
(e_1 e_2)[e/x] &= (e_1[e/x]) (e_2[e/x]) \\
(\text{rec } f(y:\tau):\tau' = e_0)[e/x] &= \text{rec } f(y:\tau):\tau' = e_0[e/x] \\
&\quad (\text{where } f, y \# e) \\
(e_1, e_2)[e/x] &= (e_1[e/x], e_2[e/x]) \\
(\text{fst } e_0)[e/x] &= \text{fst } (e_0[e/x]) \\
(\text{snd } e_0)[e/x] &= \text{snd } (e_0[e/x]) \\
(\text{let } (y, z) = e_1 \text{ in } e_2)[e/x] &= \text{let } (y, z) = e_1[e/x] \text{ in } e_2[e/x] \\
&\quad (\text{where } y, z \# e) \\
(\text{left}(e_0))[e/x] &= \text{left}(e_0[e/x]) \\
(\text{right}(e_0))[e/x] &= \text{right}(e_0[e/x]) \\
\text{case } e_0 \text{ of } \{\text{left}(y) \Rightarrow e_1 ; \text{right}(z) \Rightarrow e_2\}[e/x] &= \text{case } (e_0[e/x]) \text{ of } \{\text{left}(y) \Rightarrow e_1[e/x] ; \text{right}(z) \Rightarrow e_2[e/x]\} \\
&\quad (\text{where } y, z \# e)
\end{aligned}$$

## 2 Evaluation

$e \Downarrow v$  for  $L_{\text{Arith}}$

$$\frac{}{v \Downarrow v} \quad \frac{e_1 \Downarrow v_1 \quad e_2 \Downarrow v_2}{e_1 + e_2 \Downarrow v_1 +_{\mathbb{N}} v_2} \quad \frac{e_1 \Downarrow v_1 \quad e_2 \Downarrow v_2}{e_1 \times e_2 \Downarrow v_1 \times_{\mathbb{N}} v_2}$$

$e \Downarrow v$  for  $L_{\text{If}}$

$$\frac{e_1 \Downarrow v \quad e_2 \Downarrow v}{e_1 == e_2 \Downarrow \text{true}} \quad \frac{e_1 \Downarrow v_1 \quad e_2 \Downarrow v_2 \quad v_1 \neq v_2}{e_1 == e_2 \Downarrow \text{false}}$$

$$\frac{e \Downarrow \text{true} \quad e_1 \Downarrow v_1}{\text{if } e \text{ then } e_1 \text{ else } e_2 \Downarrow v_1} \quad \frac{e \Downarrow \text{false} \quad e_2 \Downarrow v_2}{\text{if } e \text{ then } e_1 \text{ else } e_2 \Downarrow v_2}$$

$e \Downarrow v$  for  $L_{\text{Let}}$

$$\frac{e_1 \Downarrow v_1 \quad e_2[v_1/x] \Downarrow v_2}{\text{let } x = e_1 \text{ in } e_2 \Downarrow v_2}$$

$e \Downarrow v$  for  $L_{\text{Lam}}$

$$\frac{}{\lambda x. e \Downarrow \lambda x. e} \quad \frac{e_1 \Downarrow \lambda x. e \quad e_2 \Downarrow v_2 \quad e[v_2/x] \Downarrow v}{e_1 e_2 \Downarrow v}$$

$e \Downarrow v$  for  $L_{\text{Rec}}$

$$\frac{}{\text{rec } f(x). e \Downarrow \text{rec } f(x). e} \quad \frac{e_1 \Downarrow \text{rec } f(x). e \quad e_2 \Downarrow v_2 \quad e[\text{rec } f(x). e/f, v_2/x] \Downarrow v}{e_1 e_2 \Downarrow v}$$

$e \Downarrow v$  for  $L_{\text{Data}}$

$$\frac{e_1 \Downarrow v_1 \quad e_2 \Downarrow v_2}{(e_1, e_2) \Downarrow (v_1, v_2)} \quad \frac{e \Downarrow (v_1, v_2)}{\text{fst } e \Downarrow v_1} \quad \frac{e \Downarrow (v_1, v_2)}{\text{snd } e \Downarrow v_2} \quad \frac{e_1 \Downarrow (v_1, v_2) \quad e_2[v_1/x, v_2/y] \Downarrow v}{\text{let pair } (x, y) = e_1 \text{ in } e_2 \Downarrow v}$$

$$\frac{e \Downarrow v}{\text{left}(e) \Downarrow \text{left}(v)} \quad \frac{e \Downarrow v}{\text{right}(e) \Downarrow \text{right}(v)}$$

$$\frac{e \Downarrow \text{left}(v_1) \quad e_1[v_1/x] \Downarrow v}{\text{case } e \text{ of } \{\text{left}(x) \Rightarrow e_1 ; \text{right}(y) \Rightarrow e_2\} \Downarrow v} \quad \frac{e \Downarrow \text{right}(v_2) \quad e_2[v_2/y] \Downarrow v}{\text{case } e \text{ of } \{\text{left}(x) \Rightarrow e_1 ; \text{right}(y) \Rightarrow e_2\} \Downarrow v}$$

### 3 Types

$\boxed{\Gamma \vdash e : \tau}$  for  $L_{\text{Arith}}$

$$\frac{}{\Gamma \vdash n : \text{int}} \quad \frac{\Gamma \vdash e_1 : \text{int} \quad \Gamma \vdash e_2 : \text{int}}{\Gamma \vdash e_1 + e_2 : \text{int}} \quad \frac{\Gamma \vdash e_1 : \text{int} \quad \Gamma \vdash e_2 : \text{int}}{\Gamma \vdash e_1 \times e_2 : \text{int}}$$

$\boxed{\Gamma \vdash e : \tau}$  for  $L_{\text{If}}$

$$\frac{}{\Gamma \vdash b : \text{bool}} \quad \frac{\Gamma \vdash e_1 : \tau \quad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 == e_2 : \text{bool}} \quad \frac{\Gamma \vdash e : \text{bool} \quad \Gamma \vdash e_1 : \tau \quad \Gamma \vdash e_2 : \tau}{\Gamma \vdash \text{if } e \text{ then } e_1 \text{ else } e_2 : \tau}$$

$\boxed{\Gamma \vdash e : \tau}$  for  $L_{\text{Let}}$

$$\frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau} \quad \frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma, x : \tau_1 \vdash e_2 : \tau_2}{\Gamma \vdash \text{let } x = e_1 \text{ in } e_2 : \tau_2}$$

$\boxed{\Gamma \vdash e : \tau}$  for  $L_{\text{Lam}}$

$$\frac{\Gamma, x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash \lambda x : \tau_1. e : \tau_1 \rightarrow \tau_2} \quad \frac{\Gamma \vdash e_1 : \tau_1 \rightarrow \tau_2 \quad \Gamma \vdash e_2 : \tau_1}{\Gamma \vdash e_1 e_2 : \tau_2}$$

$\boxed{\Gamma \vdash e : \tau}$  for  $L_{\text{Rec}}$

$$\frac{\Gamma, f : \tau_1 \rightarrow \tau_2, x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash \text{rec } f(x : \tau_1) : \tau_2. e : \tau_1 \rightarrow \tau_2}$$

$\boxed{\Gamma \vdash e : \tau}$  for  $L_{\text{Data}}$

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash (e_1, e_2) : \tau_1 \times \tau_2} \quad \frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \text{fst } e : \tau_1} \quad \frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \text{snd } e : \tau_2} \quad \frac{\Gamma \vdash e_1 : \tau_1 \times \tau_2 \quad \Gamma, x : \tau_1, y : \tau_2 \vdash e_2 : \tau}{\Gamma \vdash \text{let pair } (x, y) = e_1 \text{ in } e_2 : \tau}$$

$$\frac{\Gamma \vdash e : \tau_1}{\Gamma \vdash \text{left}(e) : \tau_1 + \tau_2} \quad \frac{\Gamma \vdash e : \tau_2}{\Gamma \vdash \text{right}(e) : \tau_1 + \tau_2} \quad \frac{\Gamma \vdash e : \tau_1 + \tau_2 \quad \Gamma, x : \tau_1 \vdash e_1 : \tau \quad \Gamma, y : \tau_2 \vdash e_2 : \tau}{\Gamma \vdash \text{case } e \text{ of } \{\text{left}(x) \Rightarrow e_1 ; \text{right}(y) \Rightarrow e_2\} : \tau}$$