Elements of Programming Languages

Lecture 2: Evaluation

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Overview

- Last time:
  - Concrete vs. abstract syntax
  - Programming with abstract syntax trees
  - A taste of induction over expressions
- Today:
  - Evaluation
  - A simple interpreter
  - Modeling evaluation using rules

Values

- Recall L_{Arith} expressions:

$$Expr \ni e ::= e_1 + e_2 \mid e_1 \times e_2 \mid n \in \mathbb{N}$$

- Some expressions, like 1,2,3, are special
- They have no remaining "computation" to do
- We call such expressions *values*.
- We can define a BNF grammar rule for values:

$$Value \ni v ::= n \in \mathbb{N}$$

Evaluation, informally

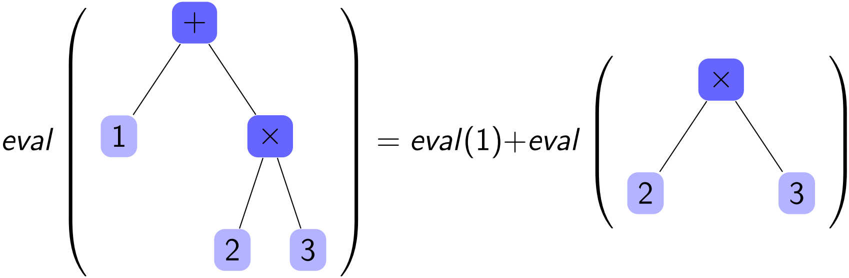
- Given an expression e, what is its value?
  - If $e = n$, a number, then it is already a value.
  - If $e = e_1 + e_2$, evaluate e_1 to v_1 and e_2 to v_2. Then add v_1 and v_2, the result is the value of e.
  - If $e = e_1 \times e_2$, evaluate e_1 to v_1 and e_2 to v_2. Then multiply v_1 and v_2, the result is the value of e.

Evaluation, in Scala

- If $e = n$, a number, then it is already a value.
- If $e = e_1 + e_2$, evaluate e_1 to v_1 and e_2 to v_2. Then add v_1 and v_2, the result is the value of e.
- If $e = e_1 \times e_2$, evaluate e_1 to v_1 and e_2 to v_2. Then multiply v_1 and v_2, the result is the value of e.

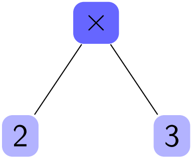
```
def eval(e: Expr): Int = e match {
  case Num(n) => n
  case Plus(e1,e2) => eval(e1) + eval(e2)
  case Times(e1,e2) => eval(e1) * eval(e2)
}
```

Example



Example

$$eval(1)+eval\left(\ \right.$$



$$\left.\ \right) = eval(1)+(eval(2)\times eval(3))$$

$$eval(1) + (eval(2) \times eval(3)) = 1 + (2 \times 3) = 1 + 6 = 7$$

Expression evaluation, more formally

- To specify and reason about evaluation, we use a *evaluation judgment*.

Definition (Evaluation judgment)

Given expression e and value v, we say v is the value of e if evaluating e results in v, and we write $e \Downarrow v$ to indicate this.

- (A *judgment* is a relation between abstract syntax trees.)
- Examples:

$$1 + 2 \Downarrow 3 \qquad 1 + 2 \times 3 \Downarrow 7 \qquad (1 + 2) \times 3 \Downarrow 9$$

Evaluation of Values

- A value is already evaluated. So, for any v, we have $v \Downarrow v$.
- We can express the fact that $v \Downarrow v$ always holds (for any v) as follows:

$$\frac{}{v \Downarrow v}$$

- This is a *rule* that says that v evaluates to v always (no preconditions)
- So, for example, we can derive:

$$\frac{}{0 \Downarrow 0} \qquad \frac{}{1 \Downarrow 1} \qquad \cdots$$

Evaluation of Addition

- How to evaluate expression $e_1 + e_2$?
- Suppose we know that $e_1 \Downarrow v_1$ and $e_2 \Downarrow v_2$.
- Then the value of $e_1 + e_2$ is the number we get by adding numbers v_1 and v_2.
- We can express this as follows:

$$\frac{e_1 \Downarrow v_1 \quad e_2 \Downarrow v_2}{e_1 + e_2 \Downarrow v_1 +_{\mathbb{N}} v_2}$$

- This is a *rule* that says that $e_1 + e_2$ evaluates to $v_1 +_{\mathbb{N}} v_2$ provided e_1 evaluates to v_1 and e_2 evaluates to v_2
- Note that we write $+_{\mathbb{N}}$ for the *mathematical function* that adds two numbers, to avoid confusion with the *abstract syntax tree* $v_1 + v_2$.

Expression evaluation: Summary

- Multiplication can be handled exactly like addition.
- We will define the meaning of L_{Arith} expressions using the following rules:

$e \Downarrow v$

$$\frac{}{v \Downarrow v} \qquad \frac{e_1 \Downarrow v_1 \quad e_2 \Downarrow v_2}{e_1 + e_2 \Downarrow v_1 +_{\mathbb{N}} v_2} \qquad \frac{e_1 \Downarrow v_1 \quad e_2 \Downarrow v_2}{e_1 \times e_2 \Downarrow v_1 \times_{\mathbb{N}} v_2}$$

- This evaluation judgment is an example of *big-step semantics* (or *natural semantics*)
  - so-called because we evaluate the whole expression "in one step"

Examples

- We can use these rules to *derive* evaluation judgments for complex expressions:

$$\dfrac{\dfrac{}{1 \Downarrow 1} \quad \dfrac{}{2 \Downarrow 2}}{1+2 \Downarrow 3} \qquad \dfrac{\dfrac{}{1 \Downarrow 1} \quad \dfrac{\dfrac{}{2 \Downarrow 2} \quad \dfrac{}{3 \Downarrow 3}}{2*3 \Downarrow 6}}{1+(2*3) \Downarrow 7} \qquad \dfrac{\dfrac{\dfrac{}{1 \Downarrow 1} \quad \dfrac{}{2 \Downarrow 2}}{1+2 \Downarrow 3} \quad \dfrac{}{3 \Downarrow 3}}{(1+2)*3 \Downarrow 9}$$

- These figures are *derivation trees* showing how we can derive a conclusion from axioms
- The rules govern how we can construct derivation trees.
  - A leaf node must match a rule with no preconditions
  - Other nodes must match rules with preconditions. (Order matters.)
- Note that derivation trees "grow up" (root is at the bottom)

Totality and Structural induction

- Question: Given any expression e, does it evaluate to a value?
- To answer this question, we can use structural induction:

Induction on structure of expressions

Given a property P of expressions, if:

- $P(n)$ holds for every number $n \in \mathbb{N}$
- for any expressions e_1, e_2, if $P(e_1)$ and $P(e_2)$ holds then $P(e_1 + e_2)$ also holds
- for any expressions e_1, e_2, if $P(e_1)$ and $P(e_2)$ holds then $P(e_1 \times e_2)$ also holds

Then $P(e)$ holds for all expressions e.

Proof by structural induction

- Let's illustrate with an example

Theorem

If e is an expression, then there exists $v \in \mathbb{N}$ such that $e \Downarrow v$ holds.

Proof: Base case.

If $e = n$ then e is already a value. Take $v = n$, then we can derive

$$\frac{}{e \Downarrow n}$$

□

Proof by structural induction

Proof: Inductive case 1.

If $e = e_1 + e_2$ then suppose $e_1 \Downarrow v_1$ and $e_2 \Downarrow v_2$ for some v_1, v_2. Then we can use the rule:

$$\frac{e_1 \Downarrow v_1 \quad e_2 \Downarrow v_2}{e_1 + e_2 \Downarrow v_1 +_{\mathbb{N}} v_2}$$

to conclude that there exists $v = v_1 +_{\mathbb{N}} v_2$ such that $e \Downarrow v$ holds. □

Note that again it's important to distinguish $v_1 +_{\mathbb{N}} v_2$ (the number) from $v_1 + v_2$ the expression.

Proof by structural induction

Proof: Inductive case 2.

If $e = e_1 \times e_2$ then suppose $e_1 \Downarrow v_1$ and $e_2 \Downarrow v_2$ for some v_1, v_2. Then we can use the rule:

$$\frac{e_1 \Downarrow v_1 \quad e_2 \Downarrow v_2}{e_1 \times e_2 \Downarrow v_1 \times_{\mathbb{N}} v_2}$$

to conclude that there exists $v = v_1 \times_{\mathbb{N}} v_2$ such that $e \Downarrow v$ holds. □

- This case is basically identical to case 1 (modulo $+$ vs. $\times$).
- From now on we will typically skip over such "essentially identical" cases (but it is important to really check them).

Uniqueness

We can also prove the uniqueness of the value of v by induction:

Theorem (Uniqueness of evaluation)

If $e \Downarrow v$ and $e \Downarrow v'$, then $v = v'$.

Base case.

If $e = n$ then since $n \Downarrow v$ and $n \Downarrow v'$ hold, the only way we could derive these judgments is for v, v' to both equal n. $\square$

Uniqueness

Inductive case.

If $e = e_1 + e_2$ then the derivations must be of the form

$$\frac{e_1 \Downarrow v_1 \quad e_2 \Downarrow v_2}{e_1 + e_2 \Downarrow v_1 +_{\mathbb{N}} v_2} \qquad \frac{e_1 \Downarrow v_1' \quad e_2 \Downarrow v_2'}{e_1 + e_2 \Downarrow v_1' +_{\mathbb{N}} v_2'}$$

By induction, $e_1 \Downarrow v_1$ and $e_1 \Downarrow v_1'$ implies $v_1 = v_1'$, and similarly for e_2 so $v_2 = v_2'$. Therefore $v_1 +_{\mathbb{N}} v_2 = v_1' +_{\mathbb{N}} v_2'$. □

- The proof for $e_1 \times e_2$ is similar.

Totality, uniqueness, and correctness

- The Scala interpreter code defined earlier says how to interpret a $\mathrm{L}_{\mathrm{Arith}}$ expression as a *function*
- The big-step rules, in contrast, specify the meaning of expressions as a *relation*.
- Nevertheless, *totality* and *uniqueness* guarantee that for each e there is a unique v such that $e \Downarrow v$
- In fact, $v = eval(e)$, that is:

Theorem (Interpreter Correctness)

For any $\mathrm{L}_{\mathrm{Arith}}$ expression e, we have $e \Downarrow v$ if and only if $v = eval(e)$.

- Proof: induction on e.

Summary

- In this lecture, we've covered:
    - A simple interpreter
    - Evaluation via rules
    - Totality and uniqueness (via structural induction)
- all for the simple language L_{Arith}
- Next time:
    - Booleans, equality, conditionals
    - Types