# Elements of Programming Languages

## Lecture 4: Variables, substitution, and scope

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September 30, 2024

# Variables

- A variable is a symbol that can ‘stand for’ a value.
- Often written $x, y, z, \ldots$.
- Let’s extend $\mathsf{L}_{\mathsf{If}}$ with variables:

$$
\begin{array}{rcl}
e & ::= & n \in \mathbb{N} \mid e_1 + e_2 \mid e_1 \times e_2 \\
 & \mid & b \in \mathbb{B} \mid e_1 == e_2 \mid \texttt{if}\ e\ \texttt{then}\ e_1\ \texttt{else}\ e_2 \\
 & \mid & x \in \mathit{Var}
\end{array}
$$

- Here, $x$ is shorthand for an arbitrary variable in $\mathit{Var}$, the set of expression variables
- Let’s call this language $\mathsf{L}_{\mathsf{Var}}$

# Aside: Operators, operators everywhere

- We have now considered several *binary operators*

$$+ \quad \times \quad \wedge \quad \vee \quad \approx$$

- as well as a unary one ($\neg$)
- It is tiresome to write their syntax, evaluation rules, and typing rules explicitly, every time we add to the language
- We will sometimes represent such operations using *schematic* syntax $e_1 \oplus e_2$ and rules:

$$\frac{e_1 \Downarrow v_1 \quad e_2 \Downarrow v_2}{e_1 \oplus e_2 \Downarrow v_1 \oplus_{\mathbb{A}} v_2} \qquad \frac{\vdash e_1 : \tau_1 \quad \vdash e_2 : \tau_2 \quad \oplus : \tau_1 \times \tau_2 \to \tau}{\vdash e_1 \oplus e_2 : \tau}$$

- where $\oplus : \tau_1 \times \tau_2 \to \tau$ means that operator $\oplus$ takes arguments $\tau_1, \tau_2$ and yields result of type $\tau$
- (e.g. $+ : \mathtt{int} \times \mathtt{int} \to \mathtt{int}$, $== : \tau \times \tau \to \mathtt{bool}$)

# Substitution

- We said "A variable can 'stand for' a value."
- What does this mean precisely?
- Suppose we have $x + 1$ and we want $x$ to "stand for" 42.
- We should be able to *replace* $x$ everywhere in $x + 1$ with 42:

$$x + 1 \rightsquigarrow 42 + 1$$

- Similarly, if $x$ "stands for" 3 then

$$\texttt{if } x == y \texttt{ then } x \texttt{ else } y \rightsquigarrow \texttt{if } 3 == y \texttt{ then } 3 \texttt{ else } y$$

# Substitution

- Let's introduce a notation for this *substitution* operation:

**Definition (Substitution)**

Given $e, x, v$, the *substitution of* $v$ *for* $x$ *in* $e$ is an expression written $e[v/x]$.

- For $\mathsf{L}_{\mathsf{Var}}$, define substitution as follows:

$$
\begin{array}{rcl}
v_0[v/x] & = & v_0 \\
x[v/x] & = & v \\
y[v/x] & = & y \quad (x \neq y) \\
(e_1 \oplus e_2)[v/x] & = & e_1[v/x] \oplus e_2[v/x] \\
(\texttt{if } e \texttt{ then } e_1 \texttt{ else } e_2)[v/x] & = & \texttt{if } e[v/x] \texttt{ then } e_1[v/x] \\
 & & \texttt{ else } e_2[v/x]
\end{array}
$$

# Scope

- As we all know from programming, we can *reuse* variable names:

```
def foo(x: Int) = x + 1
def bar(x: Int) = x * x
```

- The occurrences of `x` in `foo` have nothing to do with those in `bar`
- Moreover the following code is equivalent (since `y` is not already in use in `foo` or `bar`):

```
def foo(x: Int) = x + 1
def bar(y: Int) = y * y
```

# Scope

**Definition (Scope)**

The *scope* of a variable name is the collection of program locations in which occurrences of the variable refer to the same thing.

- I am being a little casual here: "refer to the same thing" doesn't necessarily mean that the two variable occurrences evaluate to the same value at run time.
- For example, the variables could refer to a shared *reference cell* whose value changes over time.
- In that case, the "same thing" they refer to is the reference cell, not the value in it.

# Scope, Binding and Bound Variables

- Certain occurrences of variables are called *binding*
- Again, consider

```
def foo(x: Int) = x + 1
def bar(y: Int) = y * y
```

- The occurrences of `x` and `y` on the left-hand side of the definitions are *binding*
- Binding occurrences define scopes: the occurrences of `x` and `y` on the right-hand side are *bound*
- Any variables not in scope of a binder are called *free*
- Key idea: Renaming all binding and bound occurrences in a scope *consistently* (avoiding name clashes) should not affect meaning

# Simple scope: let-binding

- For now, we consider a very basic form of scope: let-binding.

$$e ::= \cdots \mid x \mid \texttt{let } x = e_1 \texttt{ in } e_2$$

- We define $\mathrm{L}_{\mathrm{Let}}$ to be $\mathrm{L}_{\mathrm{If}}$ extended with variables and `let`.
- In an expression of the form $\texttt{let } x = e_1 \texttt{ in } e_2$, we say that $x$ is *bound* in $e_2$
- Intuition: let-binding allows us to use a variable $x$ as an abbreviation for (the value of) some other expression:

$$\texttt{let } x = 1 + 2 \texttt{ in } 4 \times x \rightsquigarrow \texttt{let } x = 3 \texttt{ in } 4 \times x \rightsquigarrow 4 \times 3$$

# Equivalence up to consistent renaming

- We wish to consider expressions *equivalent* (written $e_1 \equiv e_2$) if they have the same binding structure
- We can *rename* bound names to get equivalent expressions:

$$\texttt{let}\ x = y + z\ \texttt{in}\ x == w \equiv \texttt{let}\ u = y + z\ \texttt{in}\ u == w$$

- But some renamings change the binding structure:

$$\texttt{let}\ x = y + z\ \texttt{in}\ x == w \not\equiv \texttt{let}\ w = y + z\ \texttt{in}\ w == w$$

- Intuition: Renaming to $u$ is fine, because $u$ is not already "in use".
- But renaming to $w$ changes the binding structure, since $w$ was already "in use".

# Free variables

- The set of *free variables* of an expression is defined as:

$$\begin{aligned} FV(n) &= \emptyset \\ FV(x) &= \{x\} \\ FV(e_1 \oplus e_2) &= FV(e_1) \cup FV(e_2) \\ FV(\texttt{if } e \texttt{ then } e_1 \texttt{ else } e_2) &= FV(e) \cup FV(e_1) \cup FV(e_2) \\ FV(\texttt{let } x = e_1 \texttt{ in } e_2) &= FV(e_1) \cup (FV(e_2) - \{x\}) \end{aligned}$$

  where $X - Y$ is the set of elements of $X$ that are not in $Y$

$$\{x, y, z\} - \{y\} = \{x, z\}$$

- (Recall that $e_1 \oplus e_2$ is shorthand for several cases.)
- Examples:

$$FV(x + y) = \{x, y\} \qquad FV(\texttt{let } x = y \texttt{ in } x) = \{y\}$$
$$FV(\texttt{let } x = x + y \texttt{ in } z) = \{x, y, z\}$$

# Renaming

- We will also use the following *swapping* operation to rename variables:

$$
\begin{array}{rcl}
x(y \leftrightarrow z) & = & \begin{cases} y & \text{if } x = z \\ z & \text{if } x = y \\ x & \text{otherwise} \end{cases} \\
v(y \leftrightarrow z) & = & v \\
(e_1 \oplus e_2)(y \leftrightarrow z) & = & e_1(y \leftrightarrow z) \oplus e_2(y \leftrightarrow z) \\
(\texttt{if } e \texttt{ then } e_1 \texttt{ else } e_2)(y \leftrightarrow z) & = & \texttt{if } e(y \leftrightarrow z) \texttt{ then } e_1(y \leftrightarrow z) \\
 & & \quad \texttt{else } e_2(y \leftrightarrow z) \\
(\texttt{let } x = e_1 \texttt{ in } e_2)(y \leftrightarrow z) & = & \texttt{let } x(y \leftrightarrow z) = e_1(y \leftrightarrow z) \\
 & & \quad \texttt{in } e_2(y \leftrightarrow z)
\end{array}
$$

- Example:

$$(\texttt{let } x = y \texttt{ in } x + z)(x \leftrightarrow z) = \texttt{let } z = y \texttt{ in } z + x$$

# Alpha-conversion

- We can now define “consistent renaming”.
- Suppose $y \notin FV(e_2)$. Then we can rename a `let`-expression as follows:

$$\texttt{let } x = e_1 \texttt{ in } e_2 \rightsquigarrow_\alpha \texttt{let } y = e_1 \texttt{ in } e_2(x{\leftrightarrow}y)$$

- This is called *alpha-conversion*.
- Two expressions are *alpha-equivalent* if we can convert one to the other using alpha-conversions.

# Examples

- Examples:

$$\begin{array}{rl} & \texttt{let } x = y + z \texttt{ in } x == w \\ \rightsquigarrow_\alpha & \texttt{let } u = y + z \texttt{ in } (x == w)(x \leftrightarrow u) \\ = & \texttt{let } u = y + z \texttt{ in } x(x \leftrightarrow u) == w(x \leftrightarrow u) \\ = & \texttt{let } u = y + z \texttt{ in } u == w \end{array}$$

  since $u \notin FV(x == w)$.

- But

  $$\texttt{let } x = y + z \texttt{ in } x == w \not\rightsquigarrow_\alpha \texttt{let } w = y + z \texttt{ in } w == w$$

  because $w$ already appears in $x == w$.

# Evaluation for `let` and variables

- One approach: whenever we see $\texttt{let}\ x = e_1\ \texttt{in}\ e_2$,
  1. evaluate $e_1$ to $v_1$
  2. replace $x$ with $v_1$ in $e_2$ and evaluate that

**$e \Downarrow v$ for $L_{Let}$**

$$\frac{e_1 \Downarrow v_1 \quad e_2[v_1/x] \Downarrow v_2}{\texttt{let}\ x = e_1\ \texttt{in}\ e_2 \Downarrow v_2}$$

- Note: We always substitute values for variables, and do not need a rule for “evaluating” a variable
- This evaluation strategy is called *eager*, *strict*, or (for historical reasons) *call-by-value*
- This is a design choice. We will revisit this choice (and consider alternatives) later.

# Substitution-based interpreter

```
type Variable = String
...
case class Var(x: Variable) extends Expr
case class Let(x: Variable, e1: Expr, e2: Expr)
  extends Expr
...
def eval(e: Expr): Value = e match {
  ...
  case Let(x,e1,e2) => {
    val v = eval(e1);
    val e2vx = subst(e2,v,x);
    eval(e2vx)
  }
```

- Note: No case for `Var(x)`.

# Types and variables

- Once we add variables to our language, how does that affect typing?
- Consider

  $$\texttt{let } x = e_1 \texttt{ in } e_2$$

  When is this well-formed? What type does it have?
- Consider a variable on its own: what type does it have?
- **Different occurrences of the same variable in different scopes could have different types.**
- We need a way to *keep track of* the types of variables

# Types for variables and let, informally

- Suppose we have a way of keeping track of the types of variables (say, some kind of map or table)
- When we see a variable $x$, look up its type in the map.
- When we see a `let` $x = e_1$ `in` $e_2$, find out the type of $e_1$. Suppose that type is $\tau_1$. Add the information that $x$ has type $\tau_1$ to the map, and check $e_2$ using the augmented map.
- Note: The local information about $x$'s type should not persist beyond typechecking its scope $e_2$.

# Types for variables and let, informally

- For example:

$$\texttt{let } x = 1 \texttt{ in } x + 1$$

  is well-formed: we know that $x$ must be an `int` since it is set equal to 1, and then $x + 1$ is well-formed because $x$ is an `int` and 1 is an `int`.

- On the other hand,

$$\texttt{let } x = 1 \texttt{ in if } x \texttt{ then } 42 \texttt{ else } 17$$

  is not well-formed: we again know that $x$ must be an `int` while checking `if` $x$ `then` 42 `else` 17, but then when we check that the conditional's test $x$ is a `bool`, we find that it is actually an `int`.

# Type Environments

- We write $\Gamma$ to denote a *type environment*, or a finite map from variable names to types, often written as follows:

$$\Gamma ::= x_1 : \tau_1, \ldots, x_n : \tau_n$$

- In Scala, we can use the built-in type `ListMap[Variable,Type]` for this.
    - *hey, maybe that's why the Lab has all that stuff about `ListMaps`!*
- Moreover, we write $\Gamma(x)$ for the type of $x$ according to $\Gamma$ and $\Gamma, x : \tau$ to indicate extending $\Gamma$ with the mapping $x$ to $\tau$.

# Types for variables and let, formally

- We now generalize the ideas of well-formedness:

**Definition (Well-formedness in a context)**

We write $\Gamma \vdash e : \tau$ to indicate that $e$ is well-formed at type $\tau$ (or just "has type $\tau$") in context $\Gamma$.

- The rules for variables and let-binding are as follows:

**$\boxed{\Gamma \vdash e : \tau}$ for $L_{Let}$**

$$\frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau} \qquad \frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma, x : \tau_1 \vdash e_2 : \tau_2}{\Gamma \vdash \mathtt{let}\ x = e_1\ \mathtt{in}\ e_2 : \tau_2}$$

# Types for variables and let, formally

- We also need to generalize the $L_{If}$ rules to allow contexts:

**$\Gamma \vdash e : \tau$ for $L_{If}$**

$$\frac{}{\Gamma \vdash n : \mathtt{int}} \qquad \frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2 \quad \oplus : \tau_1 \times \tau_2 \to \tau}{\Gamma \vdash e_1 \oplus e_2 : \tau}$$

$$\frac{}{\Gamma \vdash b : \mathtt{bool}} \qquad \frac{\Gamma \vdash e : \mathtt{bool} \quad \Gamma \vdash e_1 : \tau \quad \Gamma \vdash e_2 : \tau}{\Gamma \vdash \mathtt{if}\ e\ \mathtt{then}\ e_1\ \mathtt{else}\ e_2 : \tau}$$

- This is straightforward: we just add $\Gamma$ everywhere.
- The previous rules are special cases where $\Gamma$ is empty.

# Examples, revisited

We can now typecheck as follows:

$$\dfrac{\dfrac{}{\vdash 1 : \mathtt{int}} \quad \dfrac{\dfrac{}{x : \mathtt{int} \vdash x : \mathtt{int}} \quad \dfrac{}{x : \mathtt{int} \vdash 1 : \mathtt{int}}}{x : \mathtt{int} \vdash x + 1 : \mathtt{int}}}{\vdash \mathtt{let}\ x = 1\ \mathtt{in}\ x + 1 : \mathtt{int}}$$

On the other hand:

$$\dfrac{\dfrac{}{\vdash 1 : \mathtt{int}} \quad \dfrac{x : \mathtt{int} \vdash x : \mathtt{bool} \quad \cdots}{x : \mathtt{int} \vdash \mathtt{if}\ x\ \mathtt{then}\ 42\ \mathtt{else}\ 17 : ??}}{\vdash \mathtt{let}\ x = 1\ \mathtt{in}\ \mathtt{if}\ x\ \mathtt{then}\ 42\ \mathtt{else}\ 17 : ??}$$

is not derivable because the judgment $x : \mathtt{int} \vdash x : \mathtt{bool}$ isn't.

# Summary

- Today we've covered:
  - Variables that can be substituted with values
  - Scope and binding, alpha-equivalence
  - Let-binding and how it affects typing and evaluation

  Next time:
  - Functions and function types
  - Recursion