Introduction to Databases

(INFR10080)

(Predicate Logic)

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Changelog

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Logic in general

Logics are formal languages for

- representing **what we know** about the world
- **reasoning** about this knowledge (draw conclusions from it)

Two components:

Syntax	defines the sentences in the language
Semantics	defines the **meaning** of the sentences

Used in many areas of Computer Science:

- Artificial Intelligence
- Semantic Web
- Software & Hardware verification
- **Databases**
- ... many many others

Motivation for Predicate Logic

Atomic formulas of propositional logic are **too atomic**

- statements that may be true or false
- but have no internal structure

First-order (or **predicate**) **logic** (FOL) overcomes this limitation

- atomic formulas are statements about relationships between objects

Predicates and constants

Consider the statements:

- *Mary is happy*
- *John is rich*
- *Mary and John are siblings*

In propositional logic these are just atomic propositions:

- mary-is-happy
- john-is-rich
- mary-and-john-are-siblings

In first-order logic atomic statements use **predicates**,
with constants as arguments:

- Happy(Mary)
- Rich(John)
- Sibling(Mary, John)

Variables and quantifiers

Consider the statements:

- *Someone is happy*
- *Being rich does not make one happy*

FOL predicates may have variables as arguments,
whose value may be bound by **quantifiers**:

- $\exists x \, \text{Happy}(x)$
- $\neg \forall x \, \big(\text{Rich}(x) \rightarrow \text{Happy}(x) \big)$

Syntax of FOL: terms

Countably infinite supply of

- variables : $x, y, z, \ldots$
- constants : $a, b, c, \ldots$
- predicates : $P, Q, R, \ldots$ (with associated **arities**)

Term	$t := x$	variable
	$\mid a$	constant

Syntax of FOL: formulas

Formula	$\phi := P(t_1, \ldots, t_n)$	atomic formula
	$\mid \neg\phi$	negation
	$\mid \phi \wedge \phi$	conjunction
	$\mid \phi \vee \phi$	disjunction
	$\mid \phi \rightarrow \phi$	implication
	$\mid \forall x\, \phi$	universal quantification (if x occurs **free** in ϕ)
	$\mid \exists x\, \phi$	existential quantification (if x occurs **free** in ϕ)

Quantifiers and free variables

Variables that are not in the scope of any quantifier
A variable that is not free is **bound**

Example: $\forall x \left(R(y, z) \wedge \exists y \, (\neg P(y, x) \vee R(y, z)) \right)$

Variables in blue are free, the others are bound

We assume quantifiers bind till the end of the formula:

Example: the formula above can be written as

$$\forall x \, R(y, z) \wedge \exists y \, \neg P(y, x) \vee R(y, z)$$

Notation We write $\exists x_1 \exists x_2 \cdots \exists x_n \, \phi$ as $\exists x_1, \ldots, x_n \, \phi$
and $\forall x_1 \forall x_2 \cdots \forall x_n \, \phi$ as $\forall x_1, \ldots, x_n \, \phi$

FOL interpretations

A formula may be true (or false) w.r.t. a given **interpretation** consisting of

- a semantic function $\cdot^{\mathcal{I}}$ mapping each **predicate symbol** to a **relation** (over constants) of appropriate arity

  Example: If Person is a binary predicate, $\text{Person}^{\mathcal{I}}$ could be $\{(\text{Mary}, 24),\ (\text{John}, 32),\ \ldots\}$

- a variable assignment ν mapping each **variable** to a **constant**

  Example: $\nu = \{x \mapsto 29,\ y \mapsto \text{John},\ \ldots\}$

Notation $\nu[x/a]$ is the same as ν except that $x \mapsto a$

Example: For ν above, $\nu[y/31] = \{x \mapsto 29,\ y \mapsto 31,\ \ldots\}$

Semantics of FOL

We extend ν to be the **identity over constants**

(so that we can apply ν to all terms)

$\mathcal{I}, \nu \models \phi$ means the interpretation $(\mathcal{I}, \nu)$ **satisfies** formula ϕ

$$
\begin{array}{lcl}
\mathcal{I}, \nu \models P(t_1, \ldots, t_n) & \iff & \big(\nu(t_1), \ldots, \nu(t_n)\big) \in P^{\mathcal{I}} \\
\mathcal{I}, \nu \models \neg\phi & \iff & \mathcal{I}, \nu \not\models \phi \\
\mathcal{I}, \nu \models \phi \wedge \psi & \iff & \mathcal{I}, \nu \models \phi \text{ and } \mathcal{I}, \nu \models \psi \\
\mathcal{I}, \nu \models \phi \vee \psi & \iff & \mathcal{I}, \nu \models \phi \text{ or } \mathcal{I}, \nu \models \psi \\
\mathcal{I}, \nu \models \phi \rightarrow \psi & \iff & \text{if } \mathcal{I}, \nu \models \phi \text{ then } \mathcal{I}, \nu \models \psi \\
\mathcal{I}, \nu \models \forall x\, \phi & \iff & \text{for every constant } a\text{: } \mathcal{I}, \nu[x/a] \models \phi \\
\mathcal{I}, \nu \models \exists x\, \phi & \iff & \text{there is a constant } a \text{ s.t. } \mathcal{I}, \nu[x/a] \models \phi
\end{array}
$$

Equality

Equality is a special predicate

$t_1 = t_2$ is true under a given interpretation

if and only if

t_1 and t_2 refer to the same constant

That is,

$$\mathcal{I}, \nu \models t_1 = t_2 \quad \Longleftrightarrow \quad \nu(t_1) = \nu(t_2)$$

Examples

- Let the set of constants be $\{\text{John}, \text{Mary}, \text{Jane}, \text{Scooby}\} \cup \mathbb{N}$
- Consider the predicates $\text{Person}(\cdot\,,\cdot)$ and $\text{Happy}(\,\cdot\,)$
- Take the semantic function $\mathcal{I}$ such that

$$\text{Person}^{\mathcal{I}} = \{(\text{John}, 24), (\text{Jane}, 20), (\text{Mary}, 26)\}$$
$$\text{Happy}^{\mathcal{I}} = \{\text{Scooby}, \text{Jane}, \text{Mary}\}$$

Is there an assignment ν such that $(\mathcal{I}, \nu)$ **satisfies**

- $\text{Happy}(x) \wedge \neg\exists y\, \text{Person}(x, y)$?
- $\exists x, y\, \text{Person}(x, z) \wedge \text{Person}(y, z)$?
- $\exists x, y\, \text{Person}(x, z) \wedge \text{Person}(y, z) \wedge \neg(x = y)$?
- $\forall x\, \text{Happy}(x) \rightarrow \exists y\, \text{Person}(x, y)$?

Satisfiability and validity

An interpretation $(\mathcal{I}, \nu)$ is a **model** of ϕ if $\mathcal{I}, \nu \models \phi$

A formula is

satisfiable	if it has a model
unsatisfiable	if it has no models
falsifiable	if there is some interpretation that is not a model
valid	(i.e., a tautology) if every intepretation is a model

Equivalence

Two formulas are **logically equivalent** (written $\phi \equiv \psi$)
if they have the same models

That is, for all interpretations $(\mathcal{I}, \nu)$

$$\mathcal{I}, \nu \models \phi \quad \iff \quad \mathcal{I}, \nu \models \psi$$

Questions:

- Are $P(x)$ and $P(y)$ logically equivalent?
- What about $\forall x\, P(x)$ and $\forall y\, P(y)$?

Universal quantification

Everyone taking IDB is smart:

$$\forall x \left(\text{Takes}(x, \text{idb}) \rightarrow \text{Smart}(x) \right)$$

Typically $\rightarrow$ is the main connective with $\forall$

Common mistake: using $\wedge$ as the main connective with $\forall$:

$$\forall x \left(\text{Takes}(x, \text{idb}) \wedge \text{Smart}(x) \right)$$

means “Everyone takes IDB, and everyone is smart”

Existential quantification

Someone takes IDB and fails:

$$\exists x \left(\text{Takes}(x, \text{idb}) \wedge \text{Fails}(x, \text{idb}) \right)$$

Typically $\wedge$ is the main connective with $\exists$

Common mistake: using $\rightarrow$ as the main connective with $\exists$:

$$\exists x \left(\text{Takes}(x, \text{idb}) \rightarrow \text{Fails}(x, \text{idb}) \right)$$

is true if there is anyone who does not take IDB

Properties of quantifiers

- $\forall x \forall y \, \phi$ is the same as $\forall y \forall x \, \phi$
- $\exists x \exists y \, \phi$ is the same as $\exists y \exists x \, \phi$
- $\exists x \forall y \, \phi$ is **not the same** as $\forall y \exists x \, \phi$

Example

$$\exists x \forall y \, \text{Loves}(x, y)$$

means "There is somebody who loves everyone in the world"

$$\forall y \exists x \, \text{Loves}(x, y)$$

means "Everyone is loved by somebody (not necessarily the same)"

Quantifier duality

Each can be expressed using the other:

$$\forall x \, \text{Likes}(x, \text{cake}) \equiv \neg \exists x \, \neg \text{Likes}(x, \text{cake})$$

Everybody likes cakes is the same as saying
There is not anybody who does not like cake

$$\exists x \, \text{Likes}(x, \text{broccoli}) \equiv \neg \forall x \, \neg \text{Likes}(x, \text{broccoli})$$

Somebody likes broccoli is the same as saying
Not everybody does not like broccoli

Equivalences (1)

Commutativity	$\phi \lor \psi$	$\equiv$	$\psi \lor \phi$
	$\phi \land \psi$	$\equiv$	$\psi \land \phi$
Associativity	$(\phi \lor \psi) \lor \chi$	$\equiv$	$\phi \lor (\psi \lor \chi)$
	$(\phi \land \psi) \land \chi$	$\equiv$	$\phi \land (\psi \land \chi)$
Distributivity	$\phi \land (\psi \lor \chi)$	$\equiv$	$(\phi \land \psi) \lor (\phi \land \chi)$
	$\phi \lor (\psi \land \chi)$	$\equiv$	$(\phi \lor \psi) \land (\phi \lor \chi)$
Idempotence	$\phi \lor \phi$	$\equiv$	ϕ
	$\phi \land \phi$	$\equiv$	ϕ
Absorption	$\phi \lor (\phi \land \psi)$	$\equiv$	ϕ
	$\phi \land (\phi \lor \psi)$	$\equiv$	ϕ

Equivalences (2)

Double Negation	$\neg\neg\phi$	$\equiv$	ϕ
De Morgan	$\neg(\phi \lor \psi)$	$\equiv$	$\neg\phi \land \neg\psi$
	$\neg(\phi \land \psi)$	$\equiv$	$\neg\phi \lor \neg\psi$
Implication	$\phi \to \psi$	$\equiv$	$\neg\phi \lor \psi$

Equivalences (3)

$$(\forall x\, \phi) \wedge \psi \equiv \forall x\, (\phi \wedge \psi) \qquad \text{if } x \text{ is not free in } \psi$$
$$(\forall x\, \phi) \vee \psi \equiv \forall x\, (\phi \vee \psi) \qquad \text{if } x \text{ is not free in } \psi$$
$$(\exists x\, \phi) \wedge \psi \equiv \exists x\, (\phi \wedge \psi) \qquad \text{if } x \text{ is not free in } \psi$$
$$(\exists x\, \phi) \vee \psi \equiv \exists x\, (\phi \vee \psi) \qquad \text{if } x \text{ is not free in } \psi$$
$$(\forall x\, \phi) \wedge (\forall x\, \psi) \equiv \forall x\, (\phi \wedge \psi)$$
$$(\exists x\, \phi) \vee (\exists x\, \psi) \equiv \exists x\, (\phi \vee \psi)$$

$$\neg \forall x\, \phi \equiv \exists x\, \neg \phi$$
$$\neg \exists x\, \phi \equiv \forall x\, \neg \phi$$