Introduction to Databases
Predicate Logic

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CHANGELOG

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Problem 1. Let R and S be ternary and binary predicate names, respectively, and consider the following first-order logic formulas:

$$\exists x\, R(x,y,z) \wedge \forall y\, \exists x\, S(y,x) \to S(x,y) \tag{φ_1}$$
$$\exists x\, \forall y\, \big(S(y,x) \to S(x,y)\big) \to \exists y\, R(x,y,z) \tag{φ_2}$$
$$\big(\exists x\, S(y,x) \vee S(x,y)\big) \wedge \neg\exists y\, R(x,y,z) \tag{φ_3}$$

Rewrite each of the above formulae into equivalent ones that use only existential quantification, conjunction, and negation. Use the equivalences presented in class, and indicate which equivalence is applied at each rewriting step.

Solution.

(φ_1)
$$\begin{aligned}
&\exists x\, R(x,y,z) \wedge \forall y\, \exists x\, S(y,x) \to S(x,y) && \equiv [\text{ quantifiers duality }]\\
&\exists x\, R(x,y,z) \wedge \neg\exists y\, \neg\exists x\, S(y,x) \to S(x,y) && \equiv [\text{ definition of implication }]\\
&\exists x\, R(x,y,z) \wedge \neg\exists y\, \neg\exists x\, \neg S(y,x) \vee S(x,y) && \equiv [\text{ DeMorgan's laws }]\\
&\exists x\, R(x,y,z) \wedge \neg\exists y\, \neg\exists x\, \neg\big(\neg\neg S(y,x) \wedge \neg S(x,y)\big) && \equiv [\text{ double negation elimination }]\\
&\exists x\, R(x,y,z) \wedge \neg\exists y\, \neg\exists x\, \neg\big(S(y,x) \wedge \neg S(x,y)\big)
\end{aligned}$$

(φ_2)
$$\begin{aligned}
&\exists x\, \forall y\, \big(S(y,x) \to S(x,y)\big) \to \exists y\, R(x,y,z) && \equiv [\text{ definition of implication }]\\
&\exists x\, \forall y\, \big(\neg S(y,x) \vee S(x,y)\big) \to \exists y\, R(x,y,z) && \equiv [\text{ definition of implication }]\\
&\exists x\, \forall y\, \neg\big(\neg S(y,x) \vee S(x,y)\big) \vee \exists y\, R(x,y,z) && \equiv [\text{ DeMorgan's laws }]\\
&\exists x\, \forall y\, \big(\neg\neg S(y,x) \wedge \neg S(x,y)\big) \vee \exists y\, R(x,y,z) && \equiv [\text{ double negation elimination }]\\
&\exists x\, \forall y\, \big(S(y,x) \wedge \neg S(x,y)\big) \vee \exists y\, R(x,y,z) && \equiv [\text{ duality of quantifiers }]\\
&\exists x\, \neg\exists y\, \neg\big(\big(S(y,x) \wedge \neg S(x,y)\big) \vee \exists y\, R(x,y,z)\big) && \equiv [\text{ DeMorgan's laws }]\\
&\exists x\, \neg\exists y\, \neg\big(S(y,x) \wedge \neg S(x,y)\big) \wedge \neg\exists y\, R(x,y,z)
\end{aligned}$$

(φ_3)
$$\begin{aligned}
&\big(\exists x\, S(y,x) \vee S(x,y)\big) \wedge \neg\exists y\, R(x,y,z) && \equiv [\text{ DeMorgan's laws }]\\
&\big(\exists x\, \neg\big(\neg S(y,x) \wedge \neg S(x,y)\big)\big) \wedge \neg\exists y\, R(x,y,z)
\end{aligned}$$

Problem 2. Consider the unary predicates P for PERSON and C for CAKE, and the binary predicates H for HAVE and E for EAT, with the following understanding:

- $P(x)$ means that x is a person;
- $C(x)$ means that x is a cake;
- $H(x, y)$ means that x possesses y; and
- $E(x, y)$ means that x eats y.

(a) Write a first-order logic formula that expresses the saying "No person can have a cake and eat it too" without using universal quantifiers.

(b) Rewrite the above formula into an equivalent one of the form $\forall x\, \forall y\ \varphi \to \neg\psi$, where φ and ψ are conjunctions of predicates.

Solution.

(a) We can state that "it is not the case that there exist objects x and y such that x is person and y is a cake and x possesses y and x eats y" as follows:

$$\neg\exists x, y\ P(x) \land\ C(y) \land H(x, y) \land E(x, y)$$

Observe that this is the same as writing

$$\neg\exists x\, \exists y\ P(x) \land\ C(y) \land H(x, y) \land E(x, y)$$

which is *not* equivalent to

$$\neg\exists x\, \neg\exists y\ P(x) \land\ C(y) \land H(x, y) \land E(x, y)$$

The former states that "there is no object x for which **there exists** an object y such that …", while the latter states "there is no object x for which **there is no** object y such that …". Consistently with the notation we introduced in class, a negation in front of a shorthand existential $\neg\exists x_1, \ldots, x_n\ \varphi$ applies to the sequence of existentials as a whole (that is, $\neg\exists x_1\, \exists x_2, \ldots, \exists x_n\ \varphi$), not to each individual quantifier in it.

(b) We can use the quantifier equivalences to write:

$$\neg\neg\forall x\, \neg\neg\forall y\, \neg\big(P(x) \land\ C(y) \land H(x, y) \land E(x, y)\big)$$

Notice that the parentheses are needed here because the inner negation applies to the whole subformula, not just $P(x)$ (which is indeed what we would be stating without parentheses). We can then eliminate the double negations and choose how to commute (optional) and associate the conjuncts in the parenthesised subformula. One possibility is as follows:

$$\forall x\, \forall y\, \neg\big((P(x) \land\ C(y)) \land (H(x, y) \land E(x, y))\big)$$

We then push the negation inside using the Morgan's laws:

$$\forall x\, \forall y\, \neg\big(P(x) \land\ C(y)\big) \lor \neg\big(H(x, y) \land E(x, y)\big)$$

Finally, we use the equivalence for implication to obtain the following:

$$\forall x\, \forall y\ \big(P(x) \land\ C(y)\big) \to \neg\big(H(x, y) \land E(x, y)\big)$$

which reads "for every object x and every object y, if x is a person and y is a cake, then it is not the case that x both possesses and eats y".

Problem 3. Let $\{\text{John}, \text{Mary}, \text{Jane}, \text{Scooby}\} \cup \mathbb{N}$ be the underlying set of constants. Consider the binary predicate Person and the unary predicate Happy, and take the semantic function $\mathcal{I}$ such that

$$\text{Person}^{\mathcal{I}} = \{(\text{John}, 24), (\text{Jane}, 20), (\text{Mary}, 26)\}$$
$$\text{Happy}^{\mathcal{I}} = \{\text{Scooby}, \text{Jane}, \text{Mary}\}$$

For each of the formulas below, indicate whether there is an assignment ν such that the interpretation $(\mathcal{I}, \nu)$ satisfies the formula. If yes, provide a suitable assignment; otherwise, explain why no such assignment exists.

(a) $\text{Happy}(x) \wedge \neg\exists y\ \text{Person}(x, y)$

(b) $\exists x, y\ \text{Person}(x, z) \wedge \text{Person}(y, z)$

(c) $\exists x, y\ \text{Person}(x, z) \wedge \text{Person}(y, z) \wedge \neg(x = y)$

(d) $\forall x\ \text{Happy}(x) \to \exists y\ \text{Person}(x, y)$

Solution.

(a) Let φ denote the formula $\text{Happy}(x) \wedge \neg\exists y\ \text{Person}(x, y)$. We have:

$$\begin{aligned}\mathcal{I}, \nu \models \varphi &\iff \mathcal{I}, \nu \models \text{Happy}(x) \text{ and } \mathcal{I}, \nu \models \neg\exists y\ \text{Person}(x, y) \\ &\iff \nu(x) \in \text{Happy}^{\mathcal{I}} \quad \text{and } \mathcal{I}, \nu \not\models \exists y \underbrace{\text{Person}(x, y)}_{\varphi_1}\end{aligned}$$

and $\mathcal{I}, \nu \models \exists y\ \varphi_1 \iff$ there exists a constant a such that $\mathcal{I}, \nu[y/a] \models \varphi_1$. Let $\nu' = \nu[y/a]$, then

$$\mathcal{I}, \nu' \models \varphi_1 \iff \big(\nu'(x), \nu'(y)\big) \in \text{Person}^{\mathcal{I}} \overset{(\dagger)}{\iff} \big(\nu(x), a\big) \in \text{Person}^{\mathcal{I}}$$

where $(\dagger)$ is because ν' is identical to ν except that it maps y to a, so $\nu'(x) = \nu(x)$ and $\nu'(y) = a$.

So, putting it all together, we have:

$$\mathcal{I}, \nu \models \varphi \iff \nu(x) \in \text{Happy}^{\mathcal{I}} \text{ and there does not exist a constant } a \text{ such that } \big(\nu(x), a\big) \in \text{Person}^{\mathcal{I}}$$

Now, $\nu(x)$ belongs to $\text{Happy}^{\mathcal{I}}$ only if x is mapped to one of the constants Jane, Mary, or Scooby, and there is no constant a for which (Scooby, a) belongs to $\text{Person}^{\mathcal{I}}$. Therefore, the assignment ν that maps x to Scooby is (in fact, the only one) such that $\mathcal{I}, \nu \models \varphi$.

(b) Let φ denote the formula $\text{Person}(x, z) \wedge \text{Person}(y, z)$. We have:

$$\mathcal{I}, \nu \models \exists x\, \exists y\ \varphi \iff \text{there exists a constant } a \text{ such that } \mathcal{I}, \nu[x/a] \models \exists y\ \varphi$$

Let $\nu' = \nu[x/a]$, then

$$\mathcal{I}, \nu' \models \exists y\ \varphi \iff \text{there exists a constant } b \text{ such that } \mathcal{I}, \nu'[y/a] \models \varphi$$

Let $\nu'' = \nu'[y/b]$, then

$$\begin{aligned}\mathcal{I}, \nu'' \models \varphi &\iff \mathcal{I}, \nu'' \models \text{Person}(x, z) \text{ and } \mathcal{I}, \nu'' \models \text{Person}(y, z) \\ &\iff \big(\nu''(x), \nu''(z)\big) \in \text{Person}^{\mathcal{I}} \text{ and } \big(\nu''(y), \nu''(z)\big) \in \text{Person}^{\mathcal{I}} \\ &\iff \big(a, \nu(z)\big) \in \text{Person}^{\mathcal{I}} \text{ and } \big(b, \nu(z)\big) \in \text{Person}^{\mathcal{I}}\end{aligned}$$

Putting it all together, we have:

$$\mathcal{I}, \nu \models \exists x\, \exists y\ \varphi \iff \text{there are constants } a \text{ and } b \text{ such that } \big(a, \nu(z)\big) \in \text{Person}^{\mathcal{I}} \text{ and } \big(b, \nu(z)\big) \in \text{Person}^{\mathcal{I}}$$

Observe that nothing prevents a and b to be the same constant, so any assignment ν that maps variable z to one of the constants 24, 20, or 26, is such that $(\mathcal{I}, \nu)$ satisfies the formula.

(c) The situation here is as in the previous point, but in addition $(\mathcal{I}, \nu'')$ must also satisfy the formula $\neg(x = y)$, which is the case if and only if $\mathcal{I}, \nu'' \not\models x = y$. Since $\mathcal{I}, \nu'' \models x = y$ if and only if $\nu''(x) = \nu''(y)$, we overall have that $\mathcal{I}, \nu \models \exists x, y\ \mathsf{Person}(x, z) \wedge \mathsf{Person}(y, z) \wedge \neg(x = y)$ if and only if there are constants a and b such that:

- $\big(a, \nu(z)\big) \in \mathsf{Person}^{\mathcal{I}}$, and
- $\big(b, \nu(z)\big) \in \mathsf{Person}^{\mathcal{I}}$, and
- it is not the case that $a = b$.

Thus, we are now looking for *distinct* constants a and b such that the pairs $\big(a, \nu(z)\big)$ and $\big(b, \nu(z)\big)$ both belong to $\mathsf{Person}^{\mathcal{I}}$. It is easy to see that no matter what z is mapped to, we cannot satisfy these requirements: there is no assignment ν such that $(\mathcal{I}, \nu)$ satisfies the formula.

(d) Let φ denote the formula $\mathsf{Happy}(x) \rightarrow \exists y\ \mathsf{Person}(x, y)$. We have:

$$\mathcal{I}, \nu \models \forall x\ \varphi \iff \text{for every constant } a, \text{ it holds that } \mathcal{I}, \nu[x/a] \models \varphi$$

Let $\nu' = \nu[x/a]$, then

$$\begin{aligned}
\mathcal{I}, \nu' \models \varphi &\iff \text{if } \mathcal{I}, \nu' \models \mathsf{Happy}(x), \text{ then } \mathcal{I}, \nu' \models \exists y\ \mathsf{Person}(x, y) \\
&\iff \text{if } \nu'(x) \in \mathsf{Happy}^{\mathcal{I}}, \quad \text{then there is a constant } b \text{ such that } \mathcal{I}, \nu'[y/b] \models \mathsf{Person}(x, y) \\
&\iff \text{if } a \in \mathsf{Happy}^{\mathcal{I}}, \qquad \text{then there is a constant } b \text{ such that } (a, b) \in \mathsf{Person}^{\mathcal{I}}
\end{aligned}$$

Putting it all together, we have:

$$\begin{aligned}
\mathcal{I}, \nu \models \forall x\ \varphi \iff &\text{for every constant } a, \text{ it holds that:} \\
&\text{if } a \in \mathsf{Happy}^{\mathcal{I}}, \text{ then there is a constant } b \text{ such that } (a, b) \in \mathsf{Person}^{\mathcal{I}}
\end{aligned}$$

Observe that not every pair of constants in $\mathsf{Person}^{\mathcal{I}}$ is such that the first component is a constant in $\mathsf{Happy}^{\mathcal{I}}$, so there is no assignment ν such that $(\mathcal{I}, \nu)$ satisfies the formula. Indeed, while going through all the constants, as soon as we consider Scooby, which is in $\mathsf{Happy}^{\mathcal{I}}$, there would need to exist a pair in $\mathsf{Person}^{\mathcal{I}}$ with Scooby as the first component, which is clearly not the case.