

Predicate Logic

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informatics

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Changelog

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Logic in general

Logics are formal languages for

- ▶ representing **what we know** about the world
- ▶ **reasoning** about this knowledge (draw conclusions from it)

Two components:

Syntax defines the sentences in the language

Semantics defines the **meaning** of the sentences

Used in many areas of Computer Science:

- ▶ Artificial Intelligence
- ▶ Semantic Web
- ▶ Software & Hardware verification
- ▶ **Databases**
- ▶ ... many many others

Motivation for Predicate Logic

Atomic formulas of propositional logic are **too atomic**

- ▶ statements that may be true or false
- ▶ but have **no internal structure**

First-order (or **predicate**) **logic** (FOL) overcomes this limitation

- ▶ atomic formulas are statements about **relationships between objects**

Predicates and constants

Consider the statements:

- ▶ *Mary is happy*
- ▶ *John is rich*
- ▶ *Mary and John are siblings*

In propositional logic these are just atomic propositions:

- ▶ mary-is-happy
- ▶ john-is-rich
- ▶ mary-and-john-are-siblings

In first-order logic atomic statements use **predicates**, with **constants** as arguments:

- ▶ **Happy**(Mary)
- ▶ **Rich**(John)
- ▶ **Sibling**(Mary, John)

Variables and quantifiers

Consider the statements:

- ▶ *Someone is happy*
- ▶ *Being rich does not make one happy*

FOL predicates may have **variables** as arguments, whose value may be bound by **quantifiers**:

- ▶ $\exists x \text{ Happy}(x)$
- ▶ $\neg \forall x (\text{Rich}(x) \rightarrow \text{Happy}(x))$

Syntax of FOL: terms

Countably infinite supply of

variables : $x, y, z, \dots$

constants : $a, b, c, \dots$

predicates : $P, Q, R, \dots$ (with associated **arities**)

Term	$t := x$	variable
	$ a$	constant

Syntax of FOL: formulas

Formula	$\phi := P(t_1, \dots, t_n)$	atomic formula
	$ \neg \phi$	negation
	$ \phi \wedge \phi$	conjunction
	$ \phi \vee \phi$	disjunction
	$ \phi \rightarrow \phi$	implication
	$ \forall x \phi$	universal quantification (if x occurs free in ϕ)
	$ \exists x \phi$	existential quantification (if x occurs free in ϕ)

Quantifiers and free variables

Variables that are **not in the scope of any quantifier**

A variable that is not free is **bound**

Example: $\forall x (R(y, z) \wedge \exists y (\neg P(y, x) \vee R(y, z)))$

Variables in **blue** are free, the others are bound

We assume **quantifiers bind till the end** of the formula:

Example: the formula above can be written as

$$\forall x R(y, z) \wedge \exists y \neg P(y, x) \vee R(y, z)$$

Notation

We write $\exists x_1 \exists x_2 \cdots \exists x_n \phi$ as $\exists x_1, \dots, x_n \phi$
and $\forall x_1 \forall x_2 \cdots \forall x_n \phi$ as $\forall x_1, \dots, x_n \phi$

FOL interpretations

A formula may be true (or false) w.r.t. a given **interpretation** consisting of

- ▶ a **semantic function** $\mathcal{I}$ mapping each **predicate symbol** to a **relation** (over constants) of appropriate arity

Example: If Person is a binary predicate,
Person $^{\mathcal{I}}$ could be $\{(Mary, 24), (John, 32), \dots\}$

- ▶ a **variable assignment** ν mapping each **variable** to a **constant**

Example: $\nu = \{x \mapsto 29, y \mapsto John, \dots\}$

Notation $\nu[x/a]$ is the same as ν except that $x \mapsto a$

Example: For ν above, $\nu[y/31] = \{x \mapsto 29, y \mapsto 31, \dots\}$

Semantics of FOL

We extend ν to be the **identity over constants**
(so that we can apply ν to all **terms**)

$\mathcal{I}, \nu \models \phi$ means the interpretation $(\mathcal{I}, \nu)$ **satisfies** formula ϕ

$$\mathcal{I}, \nu \models P(t_1, \dots, t_n) \iff (\nu(t_1), \dots, \nu(t_n)) \in P^{\mathcal{I}}$$

$$\mathcal{I}, \nu \models \neg\phi \iff \mathcal{I}, \nu \not\models \phi$$

$$\mathcal{I}, \nu \models \phi \wedge \psi \iff \mathcal{I}, \nu \models \phi \text{ and } \mathcal{I}, \nu \models \psi$$

$$\mathcal{I}, \nu \models \phi \vee \psi \iff \mathcal{I}, \nu \models \phi \text{ or } \mathcal{I}, \nu \models \psi$$

$$\mathcal{I}, \nu \models \phi \rightarrow \psi \iff \text{if } \mathcal{I}, \nu \models \phi \text{ then } \mathcal{I}, \nu \models \psi$$

$$\mathcal{I}, \nu \models \forall x \phi \iff \text{for every constant } a: \mathcal{I}, \nu[x/a] \models \phi$$

$$\mathcal{I}, \nu \models \exists x \phi \iff \text{there is a constant } a \text{ s.t. } \mathcal{I}, \nu[x/a] \models \phi$$

Equality

Equality is a **special** predicate

$t_1 = t_2$ is true under a given interpretation
if and only if
 t_1 and t_2 refer to the same constant

That is,

$$\mathcal{I}, \nu \models t_1 = t_2 \iff \nu(t_1) = \nu(t_2)$$

Examples

- ▶ Let the set of **constants** be $\{\text{John, Mary, Jane, Scooby}\} \cup \mathbb{N}$
- ▶ Consider the **predicates** $\text{Person}(\cdot, \cdot)$ and $\text{Happy}(\cdot)$
- ▶ Take the **semantic function** $\mathcal{I}$ such that

$$\text{Person}^{\mathcal{I}} = \{(\text{John}, 24), (\text{Jane}, 20), (\text{Mary}, 26)\}$$

$$\text{Happy}^{\mathcal{I}} = \{\text{Scooby, Jane, Mary}\}$$

Is there an **assignment** ν such that $(\mathcal{I}, \nu)$ **satisfies**

- ▶ $\text{Happy}(x) \wedge \neg \exists y \text{ Person}(x, y)$?
- ▶ $\exists x, y \text{ Person}(x, z) \wedge \text{Person}(y, z)$?
- ▶ $\exists x, y \text{ Person}(x, z) \wedge \text{Person}(y, z) \wedge \neg(x = y)$?
- ▶ $\forall x \text{ Happy}(x) \rightarrow \exists y \text{ Person}(x, y)$?

Satisfiability and validity

An interpretation $(\mathcal{I}, \nu)$ is a **model** of ϕ if $\mathcal{I}, \nu \models \phi$

A formula is

satisfiable if it has a model

unsatisfiable if it has no models

falsifiable if there is some interpretation that is not a model

valid (i.e., a **tautology**) if every interpretation is a model

Equivalence

Two formulas are **logically equivalent** (written $\phi \equiv \psi$) if they have the same models

That is, for all interpretations $(\mathcal{I}, \nu)$

$$\mathcal{I}, \nu \models \phi \iff \mathcal{I}, \nu \models \psi$$

Questions:

- Are $P(x)$ and $P(y)$ logically equivalent?
- What about $\forall x P(x)$ and $\forall y P(y)$?

Universal quantification

Everyone taking IDB is smart:

$$\forall x \left(\text{Takes}(x, \text{idb}) \rightarrow \text{Smart}(x) \right)$$

Typically $\rightarrow$ is the main connective with $\forall$

Common mistake: using $\wedge$ as the main connective with $\forall$:

$$\forall x \left(\text{Takes}(x, \text{idb}) \wedge \text{Smart}(x) \right)$$

means “Everyone takes IDB, and everyone is smart”

Existential quantification

Someone takes IDB and fails:

$$\exists x \left(\text{Takes}(x, \text{idb}) \wedge \text{Fails}(x, \text{idb}) \right)$$

Typically $\wedge$ is the main connective with $\exists$

Common mistake: using $\rightarrow$ as the main connective with $\exists$:

$$\exists x \left(\text{Takes}(x, \text{idb}) \rightarrow \text{Fails}(x, \text{idb}) \right)$$

is true if there is anyone who does not take IDB

Properties of quantifiers

- ▶ $\forall x \forall y \phi$ is the same as $\forall y \forall x \phi$
- ▶ $\exists x \exists y \phi$ is the same as $\exists y \exists x \phi$
- ▶ $\exists x \forall y \phi$ is **not the same** as $\forall y \exists x \phi$

Example

$$\exists x \forall y \text{ Loves}(x, y)$$

means “There is somebody who loves everyone in the world”

$$\forall y \exists x \text{ Loves}(x, y)$$

means “Everyone is loved by somebody (not necessarily the same)”

Quantifier duality

Each can be expressed using the other:

$$\forall x \text{ Likes}(x, \text{cake}) \equiv \neg \exists x \neg \text{Likes}(x, \text{cake})$$

Everybody likes cakes is the same as saying

There is not anybody who does not like cake

$$\exists x \text{ Likes}(x, \text{broccoli}) \equiv \neg \forall x \neg \text{Likes}(x, \text{broccoli})$$

Somebody likes broccoli is the same as saying

Not everybody does not like broccoli

Equivalences (1)

Commutativity

$$\phi \vee \psi \equiv \psi \vee \phi$$

$$\phi \wedge \psi \equiv \psi \wedge \phi$$

Associativity

$$(\phi \vee \psi) \vee \chi \equiv \phi \vee (\psi \vee \chi)$$

$$(\phi \wedge \psi) \wedge \chi \equiv \phi \wedge (\psi \wedge \chi)$$

Distributivity

$$\phi \wedge (\psi \vee \chi) \equiv (\phi \wedge \psi) \vee (\phi \wedge \chi)$$

$$\phi \vee (\psi \wedge \chi) \equiv (\phi \vee \psi) \wedge (\phi \vee \chi)$$

Idempotence

$$\phi \vee \phi \equiv \phi$$

$$\phi \wedge \phi \equiv \phi$$

Absorption

$$\phi \vee (\phi \wedge \psi) \equiv \phi$$

$$\phi \wedge (\phi \vee \psi) \equiv \phi$$

Equivalences (2)

Double Negation

$$\neg\neg\phi \equiv \phi$$

De Morgan

$$\neg(\phi \vee \psi) \equiv \neg\phi \wedge \neg\psi$$

$$\neg(\phi \wedge \psi) \equiv \neg\phi \vee \neg\psi$$

Implication

$$\phi \rightarrow \psi \equiv \neg\phi \vee \psi$$

Equivalences (3)

$$(\forall x \phi) \wedge \psi \equiv \forall x (\phi \wedge \psi) \quad \text{if } x \text{ is not free in } \psi$$

$$(\forall x \phi) \vee \psi \equiv \forall x (\phi \vee \psi) \quad \text{if } x \text{ is not free in } \psi$$

$$(\exists x \phi) \wedge \psi \equiv \exists x (\phi \wedge \psi) \quad \text{if } x \text{ is not free in } \psi$$

$$(\exists x \phi) \vee \psi \equiv \exists x (\phi \vee \psi) \quad \text{if } x \text{ is not free in } \psi$$

$$(\forall x \phi) \wedge (\forall x \psi) \equiv \forall x (\phi \wedge \psi)$$

$$(\exists x \phi) \vee (\exists x \psi) \equiv \exists x (\phi \vee \psi)$$

$$\neg \forall x \phi \equiv \exists x \neg \phi$$

$$\neg \exists x \phi \equiv \forall x \neg \phi$$