# Introduction to Modern Cryptography

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(Slides courtesy of Prof. Jonathan Katz)

Lecture 11, Part 1

# Hash Functions

# Hash Functions and Message Authentication

## Recall

- We showed how to construct a secure MAC for short, **fixed-length messages** based on any PRF/block cipher
- We extended this to a secure MAC for **arbitrary-length messages** using CBC-MAC

## Question

Can we use **hash functions** to construct a secure MAC for **arbitrary-length messages**?

# Hash Functions

**(Cryptographic) hash function**

Deterministic function mapping arbitrary length inputs to a short, fixed-length output (a digest)

**Keyed or unkeyed**

- In practice, hash functions are unkeyed
- Theoretically, need to be keyed: key is public
- Assume unkeyed hash functions for simplicity

# Collision-resistance

**Collision**

Let $\boldsymbol{H : \{0,1\}^* \to \{0,1\}^l}$ be a hash function. A **collision** is a pair of distinct inputs $\boldsymbol{x, x'}$ such that $\boldsymbol{H(x) = H(x')}$

**Collision-resistance**

$\boldsymbol{H}$ is **collision-resistant** if it is infeasible to find collision in $\boldsymbol{H}$

# Generic Hash Function Attacks

## Observation

Collisions are guaranteed to exist

## Generic Attack Complexity

- What is the best **generic** collision attack on a hash function $\boldsymbol{H : \{0,1\}^* \to \{0,1\}^l}$ ?
- If we compute $\boldsymbol{H(x_1) \dots H(x_{2^l+1})}$, we are guaranteed to find a collision (why?)
- Can we do better?

# Birthday Paradox

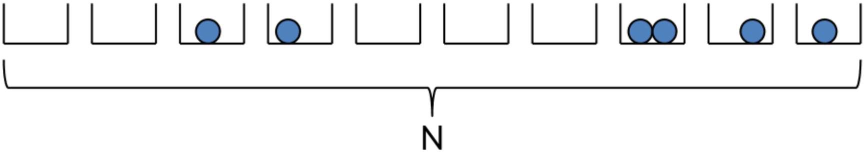
- Compute $\boldsymbol{H(x_1) \dots H(x_k)}$
- What is the probability of a collision (as a function of $\boldsymbol{k}$)?
- i.e. how many hashes do we need ($\boldsymbol{k = ?}$) in order to find a colliding pair $\boldsymbol{H(x_i) = H(x_j)}$?

**The Birthday Paradox**

How many people are needed to have a **50%** chance that some two people share a birthday?

# Birthday Paradox: Balls and Bins Experiment

How many balls do we need to throw to have a **50%** chance that two balls fall in the same bin (a collision)?




- **Bins** = days of year $\boldsymbol{N = 365}$ (#hashes $\boldsymbol{N = 2^l}$)
- **Balls** = $\boldsymbol{k}$ people ($\boldsymbol{k}$ hash function inputs)

# Birthday Attack

**Theorem**

*The collision probability is* $\mathcal{O}(k^2/N)$

- When $k \approx \sqrt{N}$, probability of a collision is $\approx 50\%$
- $k = 23$ people suffice:
- $k \approx \sqrt{2^l}$ hash-function evaluations.

**Note**

In the analysis, $H$ is modelled as a random function $\Longrightarrow$ worst case in terms of $\mathbf{Pr}$

# Security Implications of the Birthday Attack

## Implication

To protect against attackers running in time $\mathbf{2^n}$ we need the output of our hash function to be $\boldsymbol{l = 2n}$

- i.e. twice as long as symmetric keys for the same security

## Comparison to Encryption Algorithms

To protect against attackers running in time $\mathbf{2^n}$
(e.g. brute-force attack) we need the key to our symmetric-key algorithm (e.g. block-cipher keys, PRG seeds) to be $\boldsymbol{n}$

## Example

To ensure **128** bit security we need a block cipher with **128** bit key and a hash function with **256** bit output

# Birthday Bound

The birthday bound $2^{n/2}$ comes up in many other cryptographic contexts

**Example**

IV reuse in CTR-mode encryption:

- If $k$ messages are encrypted, what are the chances that some **IV** is used twice?
- Note: this is much higher than the probability that a specific **IV** is used again

# Building a Hash Function

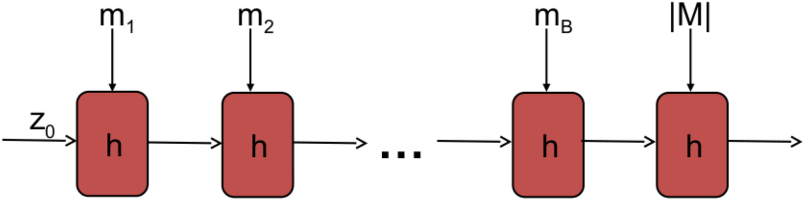
**Two-stage approach**

1. Build a **compression function $\boldsymbol{h}$** i.e. hash function for fixed-length inputs
2. Build a **hash function $\boldsymbol{H}$** for arbitrary length inputs from a compression function $\boldsymbol{h}$

# Building a Hash Function

- Assume we have a "good" compression function $\boldsymbol{h}$
  - i.e. collision-resistant for fixed-length inputs
- (Will discuss how to construct such an $\boldsymbol{h}$ later)
- Construct a hash function $\boldsymbol{H}$ (for arbitrary length inputs) based on $\boldsymbol{h}$
- Prove that collision resistance of $\boldsymbol{h}$ implies collision resistance of $\boldsymbol{H}$

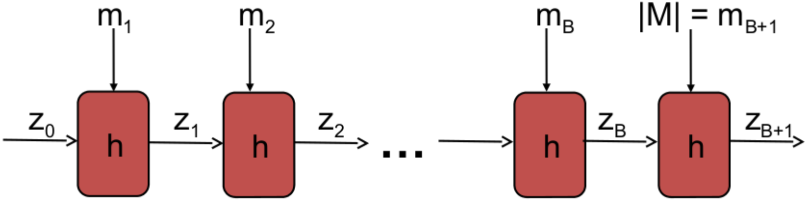
# Merkle-Damgård Transform




**Claim**

*If **h** is collision-resistant, then so is **H***

# Merkle-Damgård Transform




Proof.

Collision in $\boldsymbol{H} \implies$ collision in $\boldsymbol{h}$

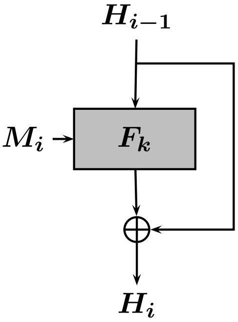
- Say $\boldsymbol{H}(\boldsymbol{m_1 \dots m_B}) = \boldsymbol{H}(\boldsymbol{m'_1 \dots m'_{B'}})$
- $|\boldsymbol{M}| \neq |\boldsymbol{M'}|$, look at the last block
- $|\boldsymbol{M}| = |\boldsymbol{M'}|$, look at largest $\boldsymbol{i}$ with $(\boldsymbol{z_{i-1}, m_i}) \neq (\boldsymbol{z'_{i-1}, m'_i})$

$\square$

# Compression Function from a PRF/Block Cipher

## Davies-Meyer

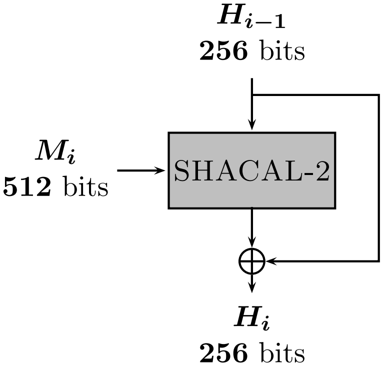
The Davies-Meyer construction is a method to transform a **block cipher** into a **compression function** using a feedforward and the message block as the key



# Example: SHA-256

## SHA-256

Merkle-Damgård + Davis-Meyer + Block cipher (SHACAL-2)



# Hash Functions in Practice

## MD5 (broken!)

- Developed in 1991
- **128**-bit output length
- Collisions found in 2004, **should no longer be used**

## SHA-1 (broken!)

- Introduced in 1995
- **160**-bit output length
- Collision found in 2017 (fixed prefix) and in 2020 (chosen prefix); **should no longer be used**

# Hash Functions in Practice

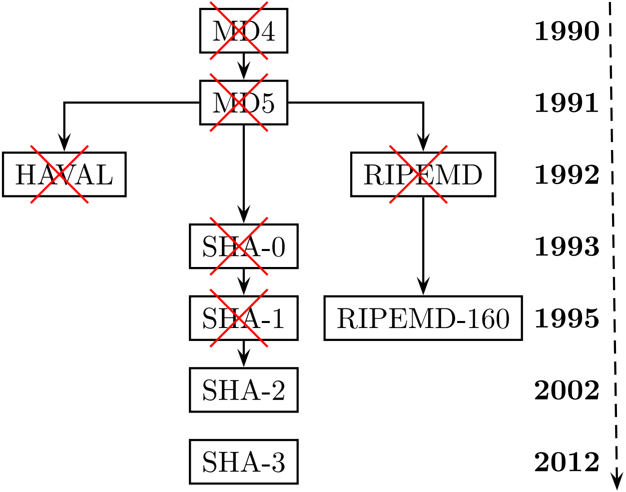
## SHA-2

- Introduced in 2001
- Versions with **224, 256, 384**, and **512**-bit outputs
- No significant known weaknesses

## SHA-3/Keccak

- Result of a public competition from 2008-2012
- Very different design than SHA-1/SHA-2
  - Does not use Merkle-Damgård transform
- Supports **224, 256, 384**, and **512**-bit outputs

# Hash Functions History




Credit: Prof. Bart Preneel

# Hash Functions and Message Authentication

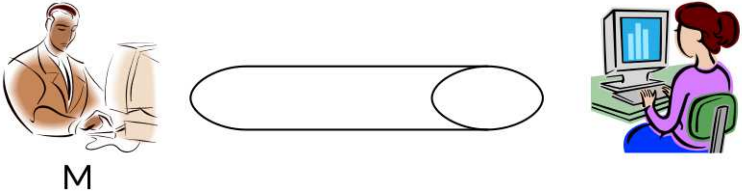
**Recall**

We showed how to construct a secure MAC for short, **fixed-length messages** based on any PRF/block cipher

**Question**

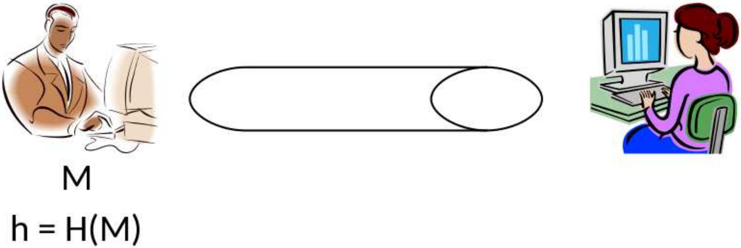
Can we use **hash functions** to construct a secure MAC for **arbitrary-length messages**?

# Main Idea



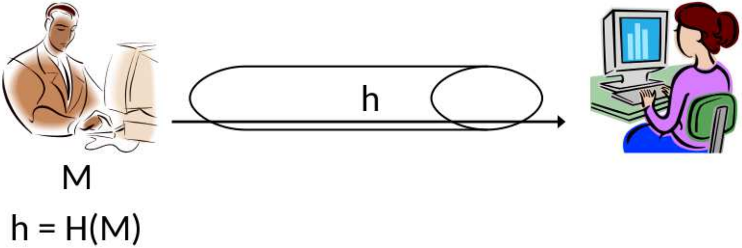

- ***A*** and ***B*** share a **reliable channel** that can handle short messages
- ***A*** wants to send reliably a **long message** ***M***

# Main Idea



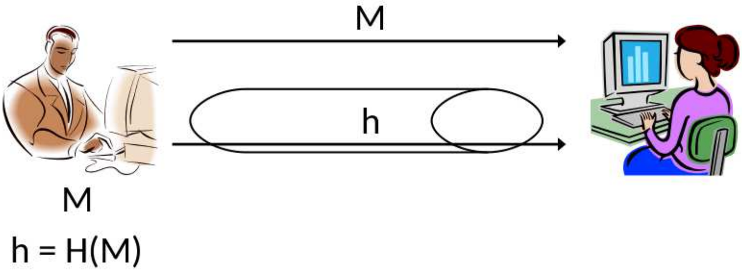

- $\boldsymbol{A}$ hashes the long message $\boldsymbol{M}$ to a shorter fixed-length digest $\boldsymbol{h = H(M)}$

# Main Idea



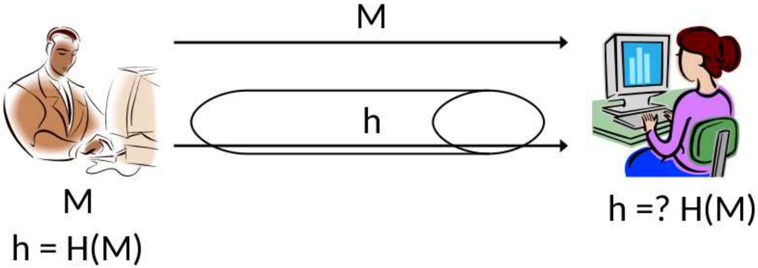

- $\boldsymbol{A}$ sends $\boldsymbol{h}$ over the reliable channel

# Main Idea




- $\boldsymbol{A}$ sends $\boldsymbol{M}$ over the general (unreliable) channel

# Main Idea




- $B$ receives $M$ and recomputes its hash $h = H(M)$
- $B$ checks whether $h$ matches the hash received by $A$
- If no match $\Longrightarrow$ the long message $M$ has been modified

# Hash-and-MAC


- $\boldsymbol{A}$ and $\boldsymbol{B}$ share a key $\boldsymbol{k}$; $\boldsymbol{A}$ transmits long message $\boldsymbol{M}$
- The reliable channel for short messages is replaced by a MAC for short messages

# Hash-and-MAC

k 

M

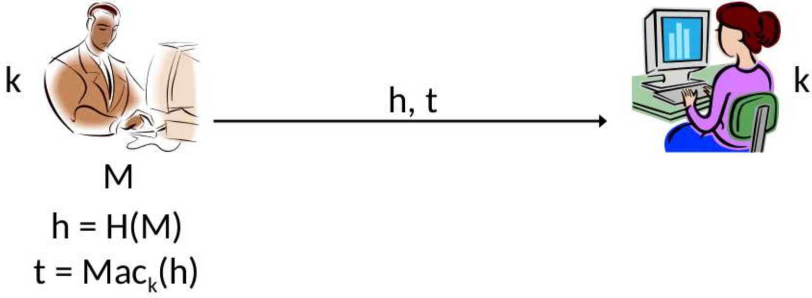
h = H(M)

t = Mac$_k$(h)

 k

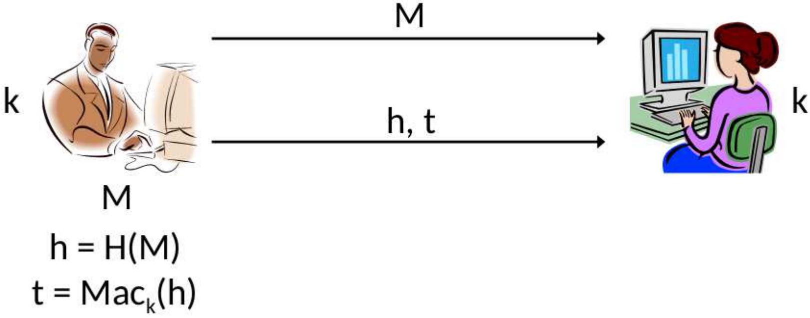
- $\boldsymbol{A}$ computes the hash $\boldsymbol{h} = \boldsymbol{H}(\boldsymbol{M})$
- $\boldsymbol{A}$ authenticates the hash with the tag $\boldsymbol{t} = \mathsf{Mac}_{\boldsymbol{k}}(\boldsymbol{h})$

# Hash-and-MAC



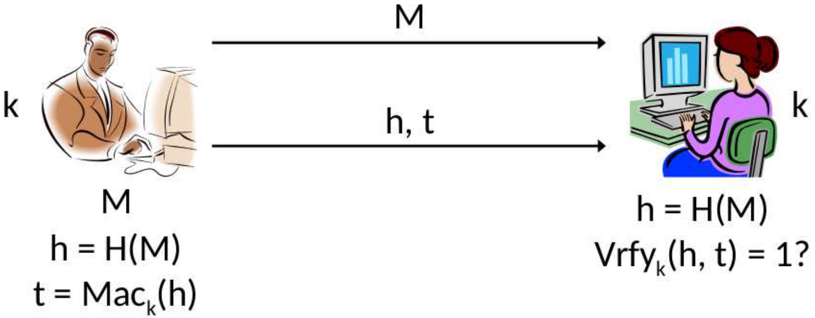

- $\boldsymbol{A}$ transmits the hash and the tag $\boldsymbol{h, t}$

# Hash-and-MAC



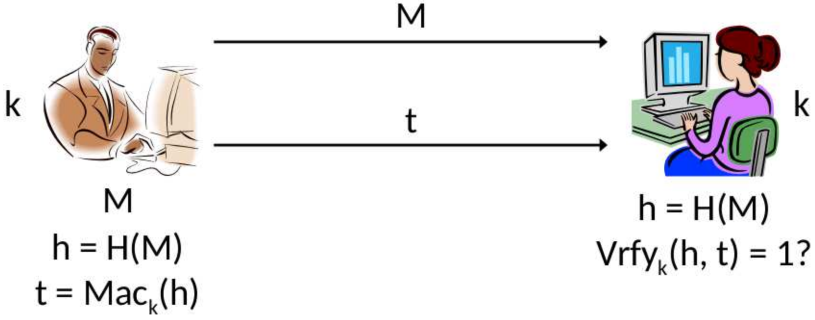

- ***A*** transmits the long message ***M***

# Hash-and-MAC




- $B$ receives $M$ and recomputes its hash $h = H(M)$
- $B$ verifies the received tag $t$ by $\mathsf{Vrfy}_k(h, t)$
- If $\mathsf{Vrfy}_k(h, t) = 1 \implies M$ has not been modified

# Hash-and-MAC




Not necessary to transmit $\boldsymbol{h}$ as $\boldsymbol{B}$ can recompute it from $\boldsymbol{M}$

# Proof of Security

## Claim

*If the MAC is secure for fixed-length messages and* $\boldsymbol{H}$ *is collision-resistant, then the [previous] construction is a secure MAC for arbitrary-length messages*

## Proof sketch

- The sender authenticates messages $\boldsymbol{M_1, M_2, \ldots}$
- As usual the attacker can choose (adaptively) $\boldsymbol{M_1, M_2, \ldots}$
- Attacker outputs forgery $\boldsymbol{(M, t) : M \neq M_i}$, $\forall \boldsymbol{i}$
- Two cases:
  1. $\boldsymbol{H(M) = H(M_i)}$ for some $\boldsymbol{i}$ $\Longrightarrow$ collision in $\boldsymbol{H}$
  2. $\boldsymbol{H(M) \neq H(M_i)}$ : $\forall \boldsymbol{i}$ $\Longrightarrow$ forgery in the underlying, fixed-length MAC

# Instantiation

**Question**

Can we instantiate the described scheme using a hash function (e.g. SHA2) and a block cipher-based MAC (e.g. AES as a PRF)?

**Problems**

- Block-length mismatch (e.g. **128** bits for AES vs. **256** bits for SHA256)
- Need to implement two crypto primitives (block cipher and hash function)

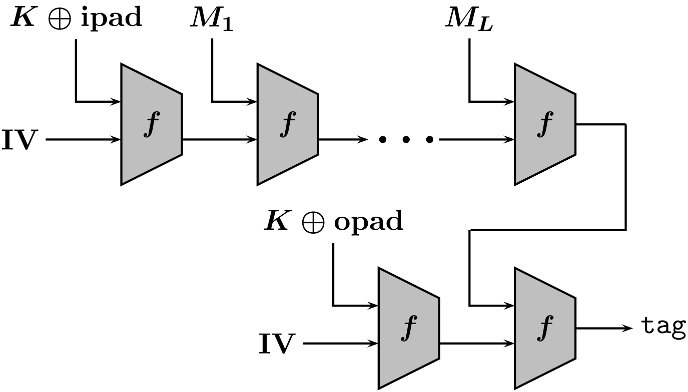
Solution: HMAC

# HMAC

HMAC is a practical instantiation of the hash-and-MAC paradigm

- Constructed entirely from Merkle-Damgård hash functions
  - MD5, SHA-1, SHA-2
  - Not SHA-3
- Follows the hash-and-MAC approach with (part of) the hash function being used as a PRF

# HMAC [Bellare,Canetti,Krawczyk,1996]




- `ipad`: inner padding (the byte 0x36 repeated $|K|$ times).
- `opad`: outer padding (the byte 0x5C repeated $|K|$ times).

# End

References: Sec 5.1, 5.2, 5.3.1, 5.4.1. Sec. 6.3 (no proofs).