Informatics 1 – Introduction to Computation
Computation and Logic

Julian Bradfield

based on materials by

Michael P. Fourman

From syllogisms
to sequent calculus

George Boole, 1815–1864

Charles Peirce, 1839–1914

Reprise

Let's summarize what we have so far, logically speaking.

- In propositional logic, we have true/false propositions $A, B, \ldots,$ and we can combine them with boolean connectives $\wedge$, $\vee$, $\neg$, $\rightarrow$.

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- We introduced sequents $a \vDash b$, which are valid iff $\forall x \in X. a(x) \rightarrow b(x)$.
- Sequents can express Aristotle's four categorical propositions: $a \vDash b$, $a \vDash \neg b$, $a \nvDash \neg b$, $a \nvDash b$.

Reprise, continued

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  - contraposition in a rule: $\dfrac{p \quad q}{r} \leftrightarrow \dfrac{p \quad \neg r}{\neg q}$

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  - contraposition in a rule: $\dfrac{p \quad\quad q}{r} \;\leftrightarrow\; \dfrac{p \quad\quad \neg r}{\neg q}$
- These gave us a modern view of Aristotle's syllogisms.

Operations on predicates

We know boolean ($\neg$, $\wedge$, $\vee$) operations on propositions.

We can lift these operations to be on *predicates*:

- $(\neg a)(x) = \neg a(x)$
- $(a \wedge b)(x) = a(x) \wedge b(x)$
- $(a \vee b)(x) = a(x) \vee b(x)$

8.—(1) Every employer shall ensure that every lifting operation involving lifting equipment is–

(a) properly planned by a competent person;

(b) appropriately supervised; and

(c) carried out in a safe manner.

The Lifting Operations and Lifting Equipment Regulations 1998

Enriching sequents (right)

The left and right of a sequent are predicates – so could be any compound predicate.

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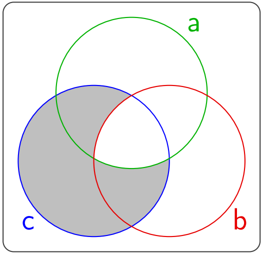
The left and right of a sequent are predicates – so could be any compound predicate. For example:

$$\frac{c \vDash a \qquad c \vDash b}{c \vDash a \wedge b}$$

Every lion is big
Every lion is fierce
∴ Every lion is big and fierce



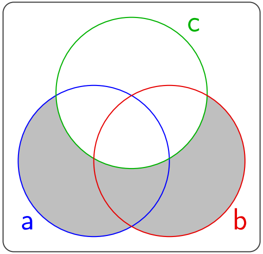
Androcles removing the thorn from the lion's paw
John Batten, in Joseph Jacobs *Europa's Fairy Book* (1916)

Enriching sequents (left)

The left and right of a sequent are predicates – so could be any compound predicate. For example:

$$\frac{a \vDash c \qquad b \vDash c}{a \vee b \vDash c}$$

Every lion is fierce
Every tiger is fierce
∴ Every lion or tiger is fierce



Male and female liger at Everland, South Korea. Wikipedia user Hkandy.

Rules for boolean combinators

We now have some rules involving each boolean combinator:

- $$\frac{a \vDash b}{\neg b \vDash \neg a}$$
- $$\frac{a \vDash c \qquad b \vDash c}{a \vee b \vDash c}$$
- $$\frac{c \vDash a \qquad c \vDash b}{c \vDash a \wedge b}$$

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Notice that we have rules with $\vee$ on the left of a sequent, and $\wedge$ on the right.

Rules for boolean combinators

We now have some rules involving each boolean combinator:

- $$\frac{a \models b}{\neg b \models \neg a}$$ (double line)
- $$\frac{a \models c \qquad b \models c}{a \vee b \models c}$$ (double line)
- $$\frac{c \models a \qquad c \models b}{c \models a \wedge b}$$ (double line)

Notice that we have rules with $\vee$ on the left of a sequent, and $\wedge$ on the right.

How should we treat $\wedge$ on the left and $\vee$ on the right?

Sets of antecedents

$$a, b, c \vDash d$$

What should this mean?

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What should this mean?

'*a*, *b*, and *c* entail *d*' ?

Sets of antecedents

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$$a \wedge b \wedge c \models d$$

$$\begin{array}{c} a \models d \qquad b \models d \\ \hline a \wedge b \models d \end{array} \quad \not\leftrightarrow \quad \begin{array}{c} a, b \models d \\ \hline\hline a \wedge b \models d \end{array}$$

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$$a \wedge b \wedge c \vDash d$$

$$\frac{a \vDash d \qquad b \vDash d}{a \wedge b \vDash d} \quad \not\leftrightarrow \quad \frac{a, b \vDash d}{a \wedge b \vDash d}$$

What's the point?

Splitting formulae into their components lets us deal with the components individually.

The empty set of antecedents

What does

$$\models d$$

mean?

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What does

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mean?

$$\models d$$

$$\varnothing \models d$$

$$\bigwedge \varnothing \models d$$

$$\top \models d$$

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$$\top \models d$$

It means d is true of everything in the universe – for short, d is true.

Sets of succedents

$$a \vDash d, e$$

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What should this mean?

$$\neg d, \neg e \vDash \neg a$$

Sets of succedents

$$a \vDash d, e$$

What should this mean?

$$\neg d, \neg e \vDash \neg a$$

$$\longleftrightarrow \quad \neg d \land \neg e \vDash \neg a$$

Sets of succedents

$$a \models d, e$$

What should this mean?

$$\begin{aligned} & \neg d, \neg e \models \neg a \\ \longleftrightarrow \quad & \neg d \land \neg e \models \neg a \\ \longleftrightarrow \quad & \neg(d \lor e) \models \neg a \end{aligned}$$

Sets of succedents

$$a \models d, e$$

What should this mean?

$$\begin{aligned} & \neg d, \neg e \models \neg a \\ \longleftrightarrow \quad & \neg d \wedge \neg e \models \neg a \\ \longleftrightarrow \quad & \neg(d \vee e) \models \neg a \\ \longleftrightarrow \quad & a \models d \vee e \end{aligned}$$

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'a entails d or e'

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'a entails d or e'

Decompose $\wedge$ on the left and $\vee$ on the right.

The empty set of succedents

What does

$$d \models$$

mean?

The empty set of succedents

What does

$$d \models$$

mean?

$$d \models$$

$$d \models \varnothing$$

$$d \models \bigvee \varnothing$$

$$d \models \bot$$

The empty set of succedents

What does

$$d \vDash$$

mean?

$$d \vDash$$

$$d \vDash \varnothing$$

$$d \vDash \bigvee \varnothing$$

$$d \vDash \bot$$

It means d is false of everything in the universe – for short, d is false.

Are you seeing a pattern between left and right of $\vDash$?

Sequents in general form

$$\Gamma \models \Delta$$

where Γ and Δ are finite *sets* of formulas (but we write them as lists for convenience).

Gerhard Gentzen
1909–1945

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$$a, b, c \vDash d, e, f$$

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'If everything in Γ holds, then something in Δ holds'

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$$\bigwedge \Gamma \vDash \bigvee \Delta$$

($\bigwedge$ is to $\wedge$ as $\bigcap$ is to $\cap$)

Gerhard Gentzen
1909–1945

The Way of the Comma

We've seen that $g, a \models b$ is the same as $g \wedge a \models b$.

And if $\Gamma = \{g_1, \ldots, g_n\}$, then $\Gamma, a \models b$ is just

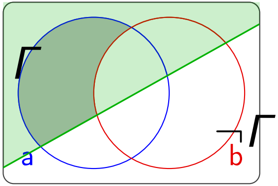
$$g_1 \wedge \cdots \wedge g_n \wedge a \models b$$

.

But it's often useful to think a bit differently:

$$\Gamma, a \models b$$

means '$a \models b$ holds in the part of the universe where Γ holds':



Comma Butterfly
Wikipedia user
Quartl

Reasoning under assumptions

Restricting to the part of the universe where Γ holds amounts to *assuming* that Γ holds, and reasoning under that assumption.

Recall the buying alcohol in Scotland example, which was formulated as a rule of legal reasoning applying just in one universe. To do it our way as sequents, define:

$A(x)$	x is over 18
$S(x)$	x is in Scotland
$D(x)$	x is between 10h and 22h†
$L(x)$	x can legally buy alcohol

The previously stated principle was

$$A, S, D \models L$$

† Note that here the universe is really the set of (*person*, *place*, *time*) triples, e.g. Seonag in Glasgow at 14:00.

From

$$A, S, D \models L$$

we can contrapone the succedent with any one antecedent:

$$A, S, \neg L \models \neg D \qquad A, \neg L, D \models \neg S \qquad \neg L, S, D \models \neg A$$

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$$A, S, D \models L$$

we can contrapone the succedent with any one antecedent:

$$A, S, \neg L \models \neg D \qquad A, \neg L, D \models \neg S \qquad \neg L, S, D \models \neg A$$

Comma is $\wedge$ – what if we contrapone *two* premises and the conclusion?

$$\begin{aligned} A, S, D \models L \quad &\longleftrightarrow \quad A, S \wedge D \models L \\ &\longleftrightarrow \quad A, \neg L \models \neg(S \wedge D) \\ &\longleftrightarrow \quad A, \neg L \models \neg S \vee \neg D \\ &\longleftrightarrow \quad A, \neg L \models \neg S, \neg D \end{aligned}$$

Can you formulate the general contraposition rule?

Introducing the sequent calculus

Gentzen's sequent calculus is (one version) of modern logical reasoning. Key differences from syllogistic reasoning:

- covers all logical formulae, not just categorical propositions
- deals only with $\models$, not with $\not\models$

We'll look at the propositional calculus: $\wedge$, $\vee$, $\neg$, but not $\forall$ and $\exists$.

Rules we've seen

$$\frac{a, b \models c}{a \wedge b \models c} \qquad \frac{c \models a \quad c \models b}{c \models a \wedge b}$$

$$\frac{a \models c \quad b \models c}{a \vee b \models c} \qquad \frac{c \models a, b}{c \models a \vee b}$$

(All these rules are also backwards sound, but we'll drop the double line to reduce clutter.)

Rules we've seen, extended

$$\frac{\Gamma, a, b \models \Delta}{\Gamma, a \land b \models \Delta} \land L \qquad \frac{\Gamma \models a, \Delta \qquad \Gamma \models b, \Delta}{\Gamma \models a \land b, \Delta} \land R$$

$$\frac{\Gamma, a \models \Delta \qquad \Gamma, b \models \Delta}{\Gamma, a \lor b \models \Delta} \lor L \qquad \frac{\Gamma \models a, b, \Delta}{\Gamma \models a \lor b, \Delta} \lor R$$

(If the rule holds in the whole universe, then it holds in the part where $\bigwedge \Gamma \land \neg \bigvee \Delta$ holds.)

Exchange

We've seen that contraposition generalizes to:

$$\frac{\Gamma \vDash a, \Delta}{\Gamma, \neg a \vDash \Delta}\ \neg L \qquad \frac{\Gamma, a \vDash \Delta}{\Gamma \vDash \neg a, \Delta}\ \neg R$$

We can take any formula, negate it and change which side it's on.

Identity

There is one other rather obvious rule we need:

$$\frac{}{\Gamma, a \vDash a, \Delta} \; I$$

(You can think of this as the base case that finishes off the long recursive call that is a proof.)

Proving statements in sequent calculus

Often we want to prove that some formula a is universally valid or a tautology – valid in every universe.

This amounts to proving

$$\vDash a$$

'a is true with no assumptions'.

For example:

$$((\neg p \vee q) \wedge \neg p) \vee p$$

is a tautology (think about it. . .).

We prove this by building a *proof tree* using the rules.

A proof!

$$\models ((\neg p \lor q) \land \neg p) \lor p$$

$$\frac{}{\Gamma, a \models a, \Delta}\ I$$

$$\frac{\Gamma \models a, \Delta}{\Gamma, \neg a \models \Delta}\ \neg L$$

$$\frac{\Gamma, a \models \Delta}{\Gamma \models \neg a, \Delta}\ \neg R$$

$$\frac{\Gamma, a, b \models \Delta}{\Gamma, a \land b \models \Delta}\ \land L$$

$$\frac{\Gamma \models a, \Delta \qquad \Gamma \models b, \Delta}{\Gamma \models a \land b, \Delta}\ \land R$$

$$\frac{\Gamma, a \models \Delta \qquad \Gamma, b \models \Delta}{\Gamma, a \lor b \models \Delta}\ \lor L$$

$$\frac{\Gamma \models a, b, \Delta}{\Gamma \models a \lor b, \Delta}\ \lor R$$

A proof!

$$\dfrac{\vDash ((\neg p \vee q) \wedge \neg p),\ \ p}{\vDash ((\neg p \vee q) \wedge \neg p) \vee p}\ \vee R$$

$$\dfrac{}{\Gamma, a \vDash a, \Delta}\ I$$

$$\dfrac{\Gamma \vDash a, \Delta}{\Gamma, \neg a \vDash \Delta}\ \neg L$$

$$\dfrac{\Gamma, a \vDash \Delta}{\Gamma \vDash \neg a, \Delta}\ \neg R$$

$$\dfrac{\Gamma, a, b \vDash \Delta}{\Gamma, a \wedge b \vDash \Delta}\ \wedge L$$

$$\dfrac{\Gamma \vDash a, \Delta \qquad \Gamma \vDash b, \Delta}{\Gamma \vDash a \wedge b, \Delta}\ \wedge R$$

$$\dfrac{\Gamma, a \vDash \Delta \qquad \Gamma, b \vDash \Delta}{\Gamma, a \vee b \vDash \Delta}\ \vee L$$

$$\dfrac{\Gamma \vDash a, b, \Delta}{\Gamma \vDash a \vee b, \Delta}\ \vee R$$

A proof!

$$\dfrac{\dfrac{\models \neg p \vee q,\ \ p \qquad\qquad \models \neg p, p}{\models ((\neg p \vee q) \wedge \neg p),\ \ p}\ \wedge R}{\models ((\neg p \vee q) \wedge \neg p) \vee p}\ \vee R$$

$$\dfrac{}{\Gamma, a \models a, \Delta}\ I$$

$$\dfrac{\Gamma \models a, \Delta}{\Gamma, \neg a \models \Delta}\ \neg L$$

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$$\dfrac{\Gamma, a, b \models \Delta}{\Gamma, a \wedge b \models \Delta}\ \wedge L$$

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$$\dfrac{\Gamma, a \models \Delta \qquad \Gamma, b \models \Delta}{\Gamma, a \vee b \models \Delta}\ \vee L$$

$$\dfrac{\Gamma \models a, b, \Delta}{\Gamma \models a \vee b, \Delta}\ \vee R$$

A proof!

$$\dfrac{\dfrac{}{\models \neg p \lor q,\ p} \qquad \dfrac{p \models p}{\models \neg p, p}\ \neg R}{\dfrac{\models ((\neg p \lor q) \land \neg p),\ p}{\models ((\neg p \lor q) \land \neg p) \lor p}\ \lor R}\ \land R$$

$$\dfrac{}{\Gamma, a \models a, \Delta}\ I$$

$$\dfrac{\Gamma \models a, \Delta}{\Gamma, \neg a \models \Delta}\ \neg L$$

$$\dfrac{\Gamma, a \models \Delta}{\Gamma \models \neg a, \Delta}\ \neg R$$

$$\dfrac{\Gamma, a, b \models \Delta}{\Gamma, a \land b \models \Delta}\ \land L$$

$$\dfrac{\Gamma \models a, \Delta \qquad \Gamma \models b, \Delta}{\Gamma \models a \land b, \Delta}\ \land R$$

$$\dfrac{\Gamma, a \models \Delta \qquad \Gamma, b \models \Delta}{\Gamma, a \lor b \models \Delta}\ \lor L$$

$$\dfrac{\Gamma \models a, b, \Delta}{\Gamma \models a \lor b, \Delta}\ \lor R$$

A proof!

$$
\dfrac{\vDash \neg p \vee q,\ p \qquad \dfrac{\dfrac{}{p \vDash p}\ I}{\vDash \neg p, p}\ \neg R}{\dfrac{\vDash ((\neg p \vee q) \wedge \neg p),\ p}{\vDash ((\neg p \vee q) \wedge \neg p) \vee p}\ \vee R}\ \wedge R
$$

$$\dfrac{}{\Gamma, a \vDash a, \Delta}\ I$$

$$\dfrac{\Gamma \vDash a, \Delta}{\Gamma, \neg a \vDash \Delta}\ \neg L$$

$$\dfrac{\Gamma, a \vDash \Delta}{\Gamma \vDash \neg a, \Delta}\ \neg R$$

$$\dfrac{\Gamma, a, b \vDash \Delta}{\Gamma, a \wedge b \vDash \Delta}\ \wedge L$$

$$\dfrac{\Gamma \vDash a, \Delta \qquad \Gamma \vDash b, \Delta}{\Gamma \vDash a \wedge b, \Delta}\ \wedge R$$

$$\dfrac{\Gamma, a \vDash \Delta \qquad \Gamma, b \vDash \Delta}{\Gamma, a \vee b \vDash \Delta}\ \vee L$$

$$\dfrac{\Gamma \vDash a, b, \Delta}{\Gamma \vDash a \vee b, \Delta}\ \vee R$$

A proof!

$$
\dfrac{\dfrac{\dfrac{\models \neg p, q, p}{\models \neg p \vee q,\ p}\ \vee R \qquad \dfrac{\dfrac{}{p \models p}\ I}{\models \neg p, p}\ \neg R}{\models ((\neg p \vee q) \wedge \neg p),\ p}\ \wedge R}{\models ((\neg p \vee q) \wedge \neg p) \vee p}\ \vee R
$$

$$\dfrac{}{\Gamma, a \models a, \Delta}\ I$$

$$\dfrac{\Gamma \models a, \Delta}{\Gamma, \neg a \models \Delta}\ \neg L$$

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$$\dfrac{\dfrac{\dfrac{p \models q, p}{\models \neg p, q, p}\;\neg R}{\models \neg p \vee q,\; p}\;\vee R \qquad \dfrac{\dfrac{}{p \models p}\;I}{\models \neg p, p}\;\neg R}{\dfrac{\models ((\neg p \vee q) \wedge \neg p),\; p}{\models ((\neg p \vee q) \wedge \neg p) \vee p}\;\vee R}\;\wedge R$$

$$\dfrac{}{\Gamma, a \models a, \Delta}\; I$$

$$\dfrac{\Gamma \models a, \Delta}{\Gamma, \neg a \models \Delta}\; \neg L$$

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$$\dfrac{\Gamma \models a, b, \Delta}{\Gamma \models a \vee b, \Delta}\; \vee R$$

A proof!

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{}{p \models q, p}\ I
    }{\models \neg p, q, p}\ \neg R
  }{\models \neg p \vee q,\ p}\ \vee R
  \qquad
  \dfrac{
    \dfrac{}{p \models p}\ I
  }{\models \neg p, p}\ \neg R
}{
  \dfrac{\models ((\neg p \vee q) \wedge \neg p),\ p}{\models ((\neg p \vee q) \wedge \neg p) \vee p}\ \vee R
}\ \wedge R
$$

$$\dfrac{}{\Gamma, a \models a, \Delta}\ I$$

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A proof!

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\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p \vDash q, p}\,I}{\vDash \neg p, q, p}\,\neg R}{\vDash \neg p \lor q,\ \ p}\,\lor R \qquad \dfrac{\dfrac{}{p \vDash p}\,I}{\vDash \neg p, p}\,\neg R}{\vDash ((\neg p \lor q) \land \neg p),\ \ p}\,\land R}{\vDash ((\neg p \lor q) \land \neg p) \lor p}\,\lor R
$$

$$\dfrac{}{\Gamma, a \vDash a, \Delta}\,I$$

$$\dfrac{\Gamma \vDash a, \Delta}{\Gamma, \neg a \vDash \Delta}\,\neg L$$

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$$\dfrac{\Gamma \vDash a, \Delta \qquad \Gamma \vDash b, \Delta}{\Gamma \vDash a \land b, \Delta}\,\land R$$

$$\dfrac{\Gamma, a \vDash \Delta \qquad \Gamma, b \vDash \Delta}{\Gamma, a \lor b \vDash \Delta}\,\lor L$$

$$\dfrac{\Gamma \vDash a, b, \Delta}{\Gamma \vDash a \lor b, \Delta}\,\lor R$$

So we have proved the formula with no assumptions.

And this was purely mechanical – we never had to think!

Finding necessary assumptions

Is $\neg((\neg a \vee b) \wedge (\neg c \vee b)) \vee (\neg a \vee c)$ a tautology?

$$\frac{}{\Gamma, a \vDash a, \Delta}\ I$$

$$\frac{\Gamma \vDash a, \Delta}{\Gamma, \neg a \vDash \Delta}\ \neg L$$

$$\frac{\Gamma, a \vDash \Delta}{\Gamma \vDash \neg a, \Delta}\ \neg R$$

$$\frac{\Gamma, a, b \vDash \Delta}{\Gamma, a \wedge b \vDash \Delta}\ \wedge L$$

$$\frac{\Gamma \vDash a, \Delta \qquad \Gamma \vDash b, \Delta}{\Gamma \vDash a \wedge b, \Delta}\ \wedge R$$

$$\frac{\Gamma, a \vDash \Delta \qquad \Gamma, b \vDash \Delta}{\Gamma, a \vee b \vDash \Delta}\ \vee L$$

$$\frac{\Gamma \vDash a, b, \Delta}{\Gamma \vDash a \vee b, \Delta}\ \vee R$$

Finding necessary assumptions

Is $\neg((\neg a \lor b) \land (\neg c \lor b)) \lor (\neg a \lor c)$ a tautology?

Try to prove it:

$$\vDash \neg((\neg a \lor b) \land (\neg c \lor b)) \lor (\neg a \lor c)$$

$$\frac{}{\Gamma, a \vDash a, \Delta}\ I$$

$$\frac{\Gamma \vDash a, \Delta}{\Gamma, \neg a \vDash \Delta}\ \neg L$$

$$\frac{\Gamma, a \vDash \Delta}{\Gamma \vDash \neg a, \Delta}\ \neg R$$

$$\frac{\Gamma, a, b \vDash \Delta}{\Gamma, a \land b \vDash \Delta}\ \land L$$

$$\frac{\Gamma \vDash a, \Delta \qquad \Gamma \vDash b, \Delta}{\Gamma \vDash a \land b, \Delta}\ \land R$$

$$\frac{\Gamma, a \vDash \Delta \qquad \Gamma, b \vDash \Delta}{\Gamma, a \lor b \vDash \Delta}\ \lor L$$

$$\frac{\Gamma \vDash a, b, \Delta}{\Gamma \vDash a \lor b, \Delta}\ \lor R$$

Finding necessary assumptions

Is $\neg((\neg a \vee b) \wedge (\neg c \vee b)) \vee (\neg a \vee c)$ a tautology?

Try to prove it:

$$\frac{\models \neg((\neg a \vee b) \wedge (\neg c \vee b)), \quad (\neg a \vee c)}{\models \neg((\neg a \vee b) \wedge (\neg c \vee b)) \vee (\neg a \vee c)} \vee R$$

$$\frac{}{\Gamma, a \models a, \Delta} I$$

$$\frac{\Gamma \models a, \Delta}{\Gamma, \neg a \models \Delta} \neg L$$

$$\frac{\Gamma, a \models \Delta}{\Gamma \models \neg a, \Delta} \neg R$$

$$\frac{\Gamma, a, b \models \Delta}{\Gamma, a \wedge b \models \Delta} \wedge L$$

$$\frac{\Gamma \models a, \Delta \qquad \Gamma \models b, \Delta}{\Gamma \models a \wedge b, \Delta} \wedge R$$

$$\frac{\Gamma, a \models \Delta \qquad \Gamma, b \models \Delta}{\Gamma, a \vee b \models \Delta} \vee L$$

$$\frac{\Gamma \models a, b, \Delta}{\Gamma \models a \vee b, \Delta} \vee R$$

Finding necessary assumptions

Is $\neg((\neg a \vee b) \wedge (\neg c \vee b)) \vee (\neg a \vee c)$ a tautology?

Try to prove it:

$$\dfrac{\dfrac{(\neg a \vee b) \wedge (\neg c \vee b) \vDash \neg a \vee c}{\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)),\quad (\neg a \vee c)}\ \neg R}{\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)) \vee (\neg a \vee c)}\ \vee R$$

$$\dfrac{}{\Gamma, a \vDash a, \Delta}\ I$$

$$\dfrac{\Gamma \vDash a, \Delta}{\Gamma, \neg a \vDash \Delta}\ \neg L$$

$$\dfrac{\Gamma, a \vDash \Delta}{\Gamma \vDash \neg a, \Delta}\ \neg R$$

$$\dfrac{\Gamma, a, b \vDash \Delta}{\Gamma, a \wedge b \vDash \Delta}\ \wedge L$$

$$\dfrac{\Gamma \vDash a, \Delta \qquad \Gamma \vDash b, \Delta}{\Gamma \vDash a \wedge b, \Delta}\ \wedge R$$

$$\dfrac{\Gamma, a \vDash \Delta \qquad \Gamma, b \vDash \Delta}{\Gamma, a \vee b \vDash \Delta}\ \vee L$$

$$\dfrac{\Gamma \vDash a, b, \Delta}{\Gamma \vDash a \vee b, \Delta}\ \vee R$$

Finding necessary assumptions

Is $\neg((\neg a \vee b) \wedge (\neg c \vee b)) \vee (\neg a \vee c)$ a tautology?

Try to prove it:

$$\dfrac{\dfrac{\dfrac{\neg a \vee b,\ \neg c \vee b \models \neg a, c}{(\neg a \vee b) \wedge (\neg c \vee b) \models \neg a \vee c}\ \wedge L, \vee R}{\models \neg((\neg a \vee b) \wedge (\neg c \vee b)),\quad (\neg a \vee c)}\ \neg R}{\models \neg((\neg a \vee b) \wedge (\neg c \vee b)) \vee (\neg a \vee c)}\ \vee R$$

$$\dfrac{}{\Gamma, a \models a, \Delta}\ I$$

$$\dfrac{\Gamma \models a, \Delta}{\Gamma, \neg a \models \Delta}\ \neg L$$

$$\dfrac{\Gamma, a \models \Delta}{\Gamma \models \neg a, \Delta}\ \neg R$$

$$\dfrac{\Gamma, a, b \models \Delta}{\Gamma, a \wedge b \models \Delta}\ \wedge L$$

$$\dfrac{\Gamma \models a, \Delta \qquad \Gamma \models b, \Delta}{\Gamma \models a \wedge b, \Delta}\ \wedge R$$

$$\dfrac{\Gamma, a \models \Delta \qquad \Gamma, b \models \Delta}{\Gamma, a \vee b \models \Delta}\ \vee L$$

$$\dfrac{\Gamma \models a, b, \Delta}{\Gamma \models a \vee b, \Delta}\ \vee R$$

Finding necessary assumptions

Is $\neg((\neg a \vee b) \wedge (\neg c \vee b)) \vee (\neg a \vee c)$ a tautology?

Try to prove it:

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg a,\ \neg c \vee b \vDash \neg a, c \qquad b,\ \neg c \vee b \vDash \neg a, c}{\neg a \vee b,\ \neg c \vee b \vDash \neg a, c}\ \vee L}{(\neg a \vee b) \wedge (\neg c \vee b) \vDash \neg a \vee c}\ \wedge L, \vee R}{\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)),\quad (\neg a \vee c)}\ \neg R}{\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)) \vee (\neg a \vee c)}\ \vee R$$

$$\dfrac{}{\Gamma, a \vDash a, \Delta}\ I$$

$$\dfrac{\Gamma \vDash a, \Delta}{\Gamma, \neg a \vDash \Delta}\ \neg L$$

$$\dfrac{\Gamma, a \vDash \Delta}{\Gamma \vDash \neg a, \Delta}\ \neg R$$

$$\dfrac{\Gamma, a, b \vDash \Delta}{\Gamma, a \wedge b \vDash \Delta}\ \wedge L$$

$$\dfrac{\Gamma \vDash a, \Delta \qquad \Gamma \vDash b, \Delta}{\Gamma \vDash a \wedge b, \Delta}\ \wedge R$$

$$\dfrac{\Gamma, a \vDash \Delta \qquad \Gamma, b \vDash \Delta}{\Gamma, a \vee b \vDash \Delta}\ \vee L$$

$$\dfrac{\Gamma \vDash a, b, \Delta}{\Gamma \vDash a \vee b, \Delta}\ \vee R$$

Finding necessary assumptions

Is $\neg((\neg a \vee b) \wedge (\neg c \vee b)) \vee (\neg a \vee c)$ a tautology?

Try to prove it:

$$
\dfrac{
\dfrac{
\dfrac{
\neg a,\ \neg c \vee b \vDash \neg a, c
\qquad
\dfrac{b, \neg c \vDash \neg a, c \qquad b, b \vDash \neg a, c}{b,\ \neg c \vee b \vDash \neg a, c}\ \vee L
}{\neg a \vee b,\ \neg c \vee b \vDash \neg a, c}\ \vee L
}{(\neg a \vee b) \wedge (\neg c \vee b) \vDash \neg a \vee c}\ \wedge L, \vee R
}{
\dfrac{\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)),\quad (\neg a \vee c)}{\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)) \vee (\neg a \vee c)}\ \vee R
}\ \neg R
$$

$$\dfrac{}{\Gamma, a \vDash a, \Delta}\ I$$

$$\dfrac{\Gamma \vDash a, \Delta}{\Gamma, \neg a \vDash \Delta}\ \neg L$$

$$\dfrac{\Gamma, a \vDash \Delta}{\Gamma \vDash \neg a, \Delta}\ \neg R$$

$$\dfrac{\Gamma, a, b \vDash \Delta}{\Gamma, a \wedge b \vDash \Delta}\ \wedge L$$

$$\dfrac{\Gamma \vDash a, \Delta \qquad \Gamma \vDash b, \Delta}{\Gamma \vDash a \wedge b, \Delta}\ \wedge R$$

$$\dfrac{\Gamma, a \vDash \Delta \qquad \Gamma, b \vDash \Delta}{\Gamma, a \vee b \vDash \Delta}\ \vee L$$

$$\dfrac{\Gamma \vDash a, b, \Delta}{\Gamma \vDash a \vee b, \Delta}\ \vee R$$

Finding necessary assumptions

Is $\neg((\neg a \vee b) \wedge (\neg c \vee b)) \vee (\neg a \vee c)$ a tautology?

Try to prove it:

$$\dfrac{a, b \vDash c}{b, b \vDash \neg a, c}\ \neg R$$

$$\dfrac{b, \neg c \vDash \neg a, c \qquad b, b \vDash \neg a, c}{b,\ \neg c \vee b \vDash \neg a, c}\ \vee L$$

$$\dfrac{\neg a,\ \neg c \vee b \vDash \neg a, c \qquad b,\ \neg c \vee b \vDash \neg a, c}{\neg a \vee b,\ \neg c \vee b \vDash \neg a, c}\ \vee L$$

$$\dfrac{\neg a \vee b,\ \neg c \vee b \vDash \neg a, c}{(\neg a \vee b) \wedge (\neg c \vee b) \vDash \neg a \vee c}\ \wedge L, \vee R$$

$$\dfrac{(\neg a \vee b) \wedge (\neg c \vee b) \vDash \neg a \vee c}{\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)),\quad (\neg a \vee c)}\ \neg R$$

$$\dfrac{\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)),\quad (\neg a \vee c)}{\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)) \vee (\neg a \vee c)}\ \vee R$$

$$\dfrac{}{\Gamma, a \vDash a, \Delta}\ I$$

$$\dfrac{\Gamma \vDash a, \Delta}{\Gamma, \neg a \vDash \Delta}\ \neg L$$

$$\dfrac{\Gamma, a \vDash \Delta}{\Gamma \vDash \neg a, \Delta}\ \neg R$$

$$\dfrac{\Gamma, a, b \vDash \Delta}{\Gamma, a \wedge b \vDash \Delta}\ \wedge L$$

$$\dfrac{\Gamma \vDash a, \Delta \qquad \Gamma \vDash b, \Delta}{\Gamma \vDash a \wedge b, \Delta}\ \wedge R$$

$$\dfrac{\Gamma, a \vDash \Delta \qquad \Gamma, b \vDash \Delta}{\Gamma, a \vee b \vDash \Delta}\ \vee L$$

$$\dfrac{\Gamma \vDash a, b, \Delta}{\Gamma \vDash a \vee b, \Delta}\ \vee R$$

Finding necessary assumptions

Is $\neg((\neg a \vee b) \wedge (\neg c \vee b)) \vee (\neg a \vee c)$ a tautology?

Try to prove it:

$$\dfrac{\neg a,\ \neg c \vee b \vDash \neg a, c \qquad \dfrac{\dfrac{a, b \vDash c}{b, \neg c \vDash \neg a, c}\ \neg R, \neg L \qquad \dfrac{a, b \vDash c}{b, b \vDash \neg a, c}\ \neg R}{b,\ \neg c \vee b \vDash \neg a, c}\ \vee L}{\dfrac{\dfrac{\neg a \vee b,\ \neg c \vee b \vDash \neg a, c}{(\neg a \vee b) \wedge (\neg c \vee b) \vDash \neg a \vee c}\ \wedge L, \vee R}{\dfrac{\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)),\quad (\neg a \vee c)}{\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)) \vee (\neg a \vee c)}\ \vee R}\ \neg R}\ \vee L$$

$$\dfrac{}{\Gamma, a \vDash a, \Delta}\ I$$

$$\dfrac{\Gamma \vDash a, \Delta}{\Gamma, \neg a \vDash \Delta}\ \neg L$$

$$\dfrac{\Gamma, a \vDash \Delta}{\Gamma \vDash \neg a, \Delta}\ \neg R$$

$$\dfrac{\Gamma, a, b \vDash \Delta}{\Gamma, a \wedge b \vDash \Delta}\ \wedge L$$

$$\dfrac{\Gamma \vDash a, \Delta \qquad \Gamma \vDash b, \Delta}{\Gamma \vDash a \wedge b, \Delta}\ \wedge R$$

$$\dfrac{\Gamma, a \vDash \Delta \qquad \Gamma, b \vDash \Delta}{\Gamma, a \vee b \vDash \Delta}\ \vee L$$

$$\dfrac{\Gamma \vDash a, b, \Delta}{\Gamma \vDash a \vee b, \Delta}\ \vee R$$

Finding necessary assumptions

Is $\neg((\neg a \vee b) \wedge (\neg c \vee b)) \vee (\neg a \vee c)$ a tautology?

Try to prove it:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg a,\ \neg c \vee b \models \neg a, c}\ I \qquad \dfrac{\dfrac{a, b \models c}{b, \neg c \models \neg a, c}\ \neg R, \neg L \qquad \dfrac{a, b \models c}{b, b \models \neg a, c}\ \neg R}{b,\ \neg c \vee b \models \neg a, c}\ \vee L}{\neg a \vee b,\ \neg c \vee b \models \neg a, c}\ \vee L}{(\neg a \vee b) \wedge (\neg c \vee b) \models \neg a \vee c}\ \wedge L, \vee R}{\models \neg((\neg a \vee b) \wedge (\neg c \vee b)),\quad (\neg a \vee c)}\ \neg R}{\models \neg((\neg a \vee b) \wedge (\neg c \vee b)) \vee (\neg a \vee c)}\ \vee R$$

$$\dfrac{}{\Gamma, a \models a, \Delta}\ I$$

$$\dfrac{\Gamma \models a, \Delta}{\Gamma, \neg a \models \Delta}\ \neg L$$

$$\dfrac{\Gamma, a \models \Delta}{\Gamma \models \neg a, \Delta}\ \neg R$$

$$\dfrac{\Gamma, a, b \models \Delta}{\Gamma, a \wedge b \models \Delta}\ \wedge L$$

$$\dfrac{\Gamma \models a, \Delta \qquad \Gamma \models b, \Delta}{\Gamma \models a \wedge b, \Delta}\ \wedge R$$

$$\dfrac{\Gamma, a \models \Delta \qquad \Gamma, b \models \Delta}{\Gamma, a \vee b \models \Delta}\ \vee L$$

$$\dfrac{\Gamma \models a, b, \Delta}{\Gamma \models a \vee b, \Delta}\ \vee R$$

Finding necessary assumptions

Is $\neg((\neg a \lor b) \land (\neg c \lor b)) \lor (\neg a \lor c)$ a tautology?

Try to prove it:

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\neg a,\ \neg c \lor b \vDash \neg a, c}\ I \qquad \dfrac{\dfrac{a, b \vDash c}{b, \neg c \vDash \neg a, c}\ \neg R, \neg L \qquad \dfrac{a, b \vDash c}{b, b \vDash \neg a, c}\ \neg R}{b,\ \neg c \lor b \vDash \neg a, c}\ \lor L}{\neg a \lor b,\ \neg c \lor b \vDash \neg a, c}\ \lor L}{(\neg a \lor b) \land (\neg c \lor b) \vDash \neg a \lor c}\ \land L, \lor R}{\dfrac{\vDash \neg((\neg a \lor b) \land (\neg c \lor b)),\quad (\neg a \lor c)}{\vDash \neg((\neg a \lor b) \land (\neg c \lor b)) \lor (\neg a \lor c)}\ \lor R}\ \neg R
$$

We are left with needing to assume $a, b \vDash c$ (or equivalently, $\vDash \neg a, \neg b, c$).

$$\dfrac{}{\Gamma, a \vDash a, \Delta}\ I$$

$$\dfrac{\Gamma \vDash a, \Delta}{\Gamma, \neg a \vDash \Delta}\ \neg L$$

$$\dfrac{\Gamma, a \vDash \Delta}{\Gamma \vDash \neg a, \Delta}\ \neg R$$

$$\dfrac{\Gamma, a, b \vDash \Delta}{\Gamma, a \land b \vDash \Delta}\ \land L$$

$$\dfrac{\Gamma \vDash a, \Delta \qquad \Gamma \vDash b, \Delta}{\Gamma \vDash a \land b, \Delta}\ \land R$$

$$\dfrac{\Gamma, a \vDash \Delta \qquad \Gamma, b \vDash \Delta}{\Gamma, a \lor b \vDash \Delta}\ \lor L$$

$$\dfrac{\Gamma \vDash a, b, \Delta}{\Gamma \vDash a \lor b, \Delta}\ \lor R$$

Assumptions provide counter-examples

The last slide showed (since all the rules work backwards) that

$$\frac{a, b \vDash c}{\neg((\neg a \vee b) \wedge (\neg c \vee b)) \vee (\neg a \vee c)}$$

So a counter-example to the conclusion is any universe where $a, b \vDash c$ fails: i.e. one where something is a and b but not c.

For example, the universe of things from week 1, taking a as 'small', b as 'triangle', and c as 'red'.

Tautologies vs non-tautologies

If you try to prove a tautology, you succeed, and every leaf of the proof tree is an I-rule.

If you try to prove a non-tautology, you get leaves with assumptions. Denying any of those assumptions gives a counter-example.

What happens when you try to prove a universally **false** statement? Try with

$$\vDash a \wedge \neg a$$