Informatics 1 – Introduction to Computation

Computation and Logic

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based on materials by

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From syllogisms
to sequent calculus
Let’s summarize what we have so far, logically speaking.

- In propositional logic, we have true/false propositions $A, B, \ldots$, and we can combine them with boolean connectives $\land, \lor, \neg, \rightarrow$. 
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- We introduced sequents $a \vdash b$, which are valid iff
  \[ \forall x \in X. a(x) \rightarrow b(x). \]

- Sequents can express Aristotle’s four categorical propositions:
  \[ a \vdash b, \ a \vdash \neg b, \ a \not\vdash \neg b, \ a \not\vdash b. \]
Reprise, continued

- We developed rules for getting new valid sequents from old ones:
Reprise, continued

- We developed rules for getting new valid sequents from old ones:
  - *barbara* 

\[
\frac{a \vdash b \quad b \vdash c}{a \vdash c}
\]
We developed rules for getting new valid sequents from old ones:

- **barbara**
  \[
  \frac{a \models b \quad b \models c}{a \models c}
  \]

- double negation
  \[\neg\neg a \leftrightarrow a\]
We developed rules for getting new valid sequents from old ones:

- **barbara** \[ \frac{a \vdash b \quad b \vdash c}{a \vdash c} \]
- double negation \[ \neg\neg a \leftrightarrow a \]
- contraposition in a sequent \[ a \vdash b \leftrightarrow \neg b \vdash \neg a \]
We developed rules for getting new valid sequents from old ones:

- **barbara**

  \[
  \begin{array}{c}
  a \vdash b \\
  b \vdash c
  \end{array} \quad \Rightarrow \quad
  \begin{array}{c}
  a \vdash c
  \end{array}
  \]

- **double negation**

  \[\neg \neg a \leftrightarrow a\]

- **contraposition in a sequent**

  \[a \vdash b \leftrightarrow \neg b \vdash \neg a\]

- **contraposition in a rule**

  \[
  \begin{array}{c}
  p \\
  q
  \end{array} \leftrightarrow 
  \begin{array}{c}
  p \\
  \neg r
  \end{array} \leftrightarrow 
  \begin{array}{c}
  \neg q
  \end{array}
  \]
We developed rules for getting new valid sequents from old ones:

- **barbara**
  \[
  \begin{array}{c}
  a \vdash b \\
  b \vdash c \\
  \hline
  a \vdash c
  \end{array}
  \]

- double negation \( \neg \neg a \leftrightarrow a \)

- contraposition in a sequent \( a \vdash b \leftrightarrow \neg b \vdash \neg a \)

- contraposition in a rule:
  \[
  \begin{array}{c}
  p \\
  q \\
  \hline
  r
  \end{array} \leftrightarrow \begin{array}{c}
  p \\
  \neg r \\
  \hline
  \neg q
  \end{array}
  \]

These gave us a modern view of Aristotle’s syllogisms.
Operations on predicates

We know boolean ($\neg$, $\land$, $\lor$) operations on propositions. We can lift these operations to be on predicates:

- $(\neg a)(x) = \neg a(x)$
- $(a \land b)(x) = a(x) \land b(x)$
- $(a \lor b)(x) = a(x) \lor b(x)$

8.—(1) Every employer shall ensure that every lifting operation involving lifting equipment is—
(a) properly planned by a competent person;
(b) appropriately supervised; and
(c) carried out in a safe manner.

_The Lifting Operations and Lifting Equipment Regulations 1998_
Enriching sequents (right)

The left and right of a sequent are predicates – so could be any compound predicate.
Enriching sequents (right)

The left and right of a sequent are predicates – so could be any compound predicate. For example:

\[ c \vdash a \quad c \vdash b \]

\[ \frac{c \vdash a \quad c \vdash b}{c \vdash a \land b} \]

Every lion is big
Every lion is fierce
∴ Every lion is big and fierce

Androcles removing the thorn from the lion’s paw
John Batten, in Joseph Jacobs
Europa’s Fairy Book (1916)
Enriching sequents (left)

The left and right of a sequent are predicates – so could be any compound predicate. For example:

\[ a \models c \quad b \models c \]

\[ a \lor b \models c \]

::: Every lion or tiger is fierce

Male and female liger at Everland, South Korea.

Wikipedia user Hkandy.
We now have some rules involving each boolean combinator:

\[
\begin{align*}
\text{If } a \vdash b, \quad &\text{then } \neg b \vdash \neg a \\
\text{If } a \vdash c, \quad b \vdash c, \quad &\text{then } \quad a \lor b \vdash c \\
\text{If } c \vdash a, \quad c \vdash b, \quad &\text{then } \quad c \vdash a \land b
\end{align*}
\]
We now have some rules involving each boolean combinator:

\[
\begin{align*}
& a \vdash b \\
\therefore & -b \vdash -a \\
& a \vdash c \quad b \vdash c \\
\therefore & a \lor b \vdash c \\
& c \vdash a \quad c \vdash b \\
\therefore & c \vdash a \land b
\end{align*}
\]

Notice that we have rules with \( \lor \) on the left of a sequent, and \( \land \) on the right.
We now have some rules involving each boolean combinator:

\[
\begin{align*}
    a & \vdash b \\
    \therefore \quad \lnot b & \vdash \lnot a
\end{align*}
\]

\[
\begin{align*}
    a & \vdash c \\
    b & \vdash c \\
    \therefore \quad a \lor b & \vdash c
\end{align*}
\]

\[
\begin{align*}
    c & \vdash a \\
    c & \vdash b \\
    \therefore \quad c & \vdash a \land b
\end{align*}
\]

Notice that we have rules with $\lor$ on the left of a sequent, and $\land$ on the right.

How should we treat $\land$ on the left and $\lor$ on the right?
Sets of antecedents

\[ a, b, c \models d \]

What should this mean?
Sets of antecedents

\[ a, b, c \models d \]

What should this mean?

‘\( a, b, \) and \( c \) entail \( d \)’?
Sets of antecedents

\[ a, b, c \models d \]

What should this mean?
‘a, b, and c entail d’ ?

\[ a \land b \land c \models d \]
Sets of antecedents

\[ a, b, c \models d \]

What should this mean?
‘a, b, and c entail d’?

\[ a \land b \land c \models d \]

\[
\begin{align*}
\frac{a \models d \quad b \models d}{a \land b \models d} & \quad \text{and} \\
\frac{a, b \models d}{a \land b \models d}
\end{align*}
\]
Sets of antecedents

\[ a, b, c \models d \]

What should this mean?
‘\(a, b, \text{ and } c \text{ entail } d\)’?

\[ a \land b \land c \models d \]

\[
\begin{align*}
&a \models d \\
&b \models d \\
\hline
&a \land b \models d
\end{align*}
\]

\[
\begin{align*}
&a, b \models d \\
\hline
&a \land b \models d
\end{align*}
\]

What’s the point?
Splitting formulae into their components lets us deal with the components individually.
The empty set of antecedents

What does $\models d$ mean?
The empty set of antecedents

What does $\models d$ mean?

$\models d$

$\emptyset \models d$

$\bigwedge \emptyset \models d$

$\top \models d$
The empty set of antecedents

What does \( \models d \) mean?

\[
\models d \\
\emptyset \models d \\
\bigwedge \emptyset \models d \\
\top \models d
\]

It means \( d \) is true of everything in the universe – for short, \( d \) is true.
Sets of succedents

\[ a \models d, e \]

What should this mean?
Sets of succedents

\[ a \vDash d, e \]

What should this mean?

\[ \neg d, \neg e \vDash \neg a \]
Sets of succedents

\[ a \models d, e \]

What should this mean?

\[ \neg d, \neg e \models \neg a \]

\[ \iff \quad \neg d \land \neg e \models \neg a \]
Sets of succedents

\[ a \models d, e \]

What should this mean?

\[ \neg d, \neg e \models \neg a \]

\[ \iff \neg (d \lor e) \models \neg a \]
Sets of succedents

\( a \models d, e \)

What should this mean?

\( \neg d, \neg e \models \neg a \)

\[ \iff \neg d \land \neg e \models \neg a \]

\[ \iff \neg (d \lor e) \models \neg a \]

\[ \iff a \models d \lor e \]
Sets of succedents

\[ a \models d, e \]

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\[ \iff a \models d \lor e \]

\[ \iff a \models d, e \]
Sets of succedents

\[ a \models d, e \]

What should this mean?

\[ \neg d, \neg e \models \neg a \]

\[ \iff \quad \neg d \land \neg e \models \neg a \]

\[ \iff \quad \neg (d \lor e) \models \neg a \]

\[ \iff \quad a \models d \lor e \]

\[ \iff \quad a \models d, e \]

‘\( a \) entails \( d \) or \( e \)’
Sets of succedents

\[ a \models d, e \]

What should this mean?

\[ \neg d, \neg e \models \neg a \]

\[ \iff \neg d \land \neg e \models \neg a \]

\[ \iff \neg (d \lor e) \models \neg a \]

\[ \iff a \models d \lor e \]

\[ \iff a \models d, e \]

‘\( a \) entails \( d \) or \( e \)’

Decompose \( \land \) on the left and \( \lor \) on the right.
What does $d \models$ mean?
The empty set of succedents

What does $d \models \emptyset$ mean?

$d \models \emptyset$

$d \models \emptyset$

$d \models \emptyset$

$d \models \emptyset$

$d \models \bot$
The empty set of succedents

What does
\[ d \models \]
mean?

\[ d \models \]
\[ d \models \emptyset \]
\[ d \models \lor \emptyset \]
\[ d \models \bot \]

It means \( d \) is false of everything in the universe – for short, \( d \) is false.

Are you seeing a pattern between left and right of \( \models \)?
Sequents in general form

\[ \Gamma \vdash \Delta \]

where \( \Gamma \) and \( \Delta \) are finite sets of formulas (but we write them as lists for convenience).
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\[ a, b, c \vdash d, e, f \]
Sequents in general form

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where \( \Gamma \) and \( \Delta \) are finite sets of formulas (but we write them as lists for convenience).

\[ a, b, c \vdash d, e, f \]

‘If everything in \( \Gamma \) holds, then something in \( \Delta \) holds’
Sequents in general form

\[ \Gamma \vDash \Delta \]

where \( \Gamma \) and \( \Delta \) are finite sets of formulas (but we write them as lists for convenience).

\[ a, b, c \vDash d, e, f \]

‘If everything in \( \Gamma \) holds, then something in \( \Delta \) holds’

\[ \bigwedge \Gamma \vDash \bigvee \Delta \]

(\( \bigwedge \) is to \( \wedge \) as \( \bigcap \) is to \( \cap \))
We’ve seen that $g, a \models b$ is the same as $g \land a \models b$.

And if $\Gamma = \{g_1, \ldots, g_n\}$, then $\Gamma, a \models b$ is just

$$g_1 \land \cdots \land g_n \land a \models b$$

But it’s often useful to think a bit differently:

$$\Gamma, a \models b$$

means ‘$a \models b$ holds in the part of the universe where $\Gamma$ holds’.
Reasoning under assumptions

Restricting to the part of the universe where $\Gamma$ holds amounts to assuming that $\Gamma$ holds, and reasoning under that assumption.

Recall the buying alcohol in Scotland example, which was formulated as a rule of legal reasoning applying just in one universe. To do it our way as sequents, define:

\[
\begin{align*}
A(x) & \quad x \text{ is over 18} \\
S(x) & \quad x \text{ is in Scotland} \\
D(x) & \quad x \text{ is between 10h and 22h} \\
L(x) & \quad x \text{ can legally buy alcohol}
\end{align*}
\]

The previously stated principle was

\[A, S, D \models L\]

† Note that here the universe is really the set of \((\text{person}, \text{place}, \text{time})\) triples, e.g. Seonag in Glasgow at 14:00.
From

$$A, S, D \models L$$

we can contrapone the succedent with any one antecedent:

$$A, S, \neg L \models \neg D \quad A, \neg L, D \models \neg S \quad \neg L, S, D \models \neg A$$
From

\[ A, S, D \models L \]

we can contrapone the succedent with any one antecedent:

\[ A, S, \neg L \models \neg D \quad A, \neg L, D \models \neg S \quad \neg L, S, D \models \neg A \]

Comma is \( \land \) – what if we contrapone two premises and the conclusion?

\[ A, S, D \models L \quad \iff \quad A, S \land D \models L \]
\[ \iff \quad A, \neg L \models \neg (S \land D) \]
\[ \iff \quad A, \neg L \models \neg S \lor \neg D \]
\[ \iff \quad A, \neg L \models \neg S, \neg D \]

Can you formulate the general contraposition rule?
Introducing the sequent calculus

Gentzen’s sequent calculus is (one version) of modern logical reasoning. Key differences from syllogistic reasoning:

- covers all logical formulae, not just categorical propositions
- deals only with $\vdash$, not with $\not\vdash$

We’ll look at the propositional calculus: $\land$, $\lor$, $\neg$, but not $\forall$ and $\exists$. 
Rules we’ve seen

\[
\begin{align*}
\frac{a, b \models c}{a \land b \models c} & \quad \frac{c \models a \quad c \models b}{c \models a \land b} \\
\frac{a \models c \quad b \models c}{a \lor b \models c} & \quad \frac{c \models a, b}{c \models a \lor b}
\end{align*}
\]

(All these rules are also backwards sound, but we’ll drop the double line to reduce clutter.)
Rules we’ve seen, extended

\[
\frac{\Gamma, a, b \models \Delta}{\Gamma, a \land b \models \Delta} \quad \land L
\]

\[
\frac{\Gamma \models a, \Delta \quad \Gamma \models b, \Delta}{\Gamma \models a \land b, \Delta} \quad \land R
\]

\[
\frac{\Gamma, a \models \Delta \quad \Gamma, b \models \Delta}{\Gamma, a \lor b \models \Delta} \quad \lor L
\]

\[
\frac{\Gamma \models a, b, \Delta}{\Gamma \models a \lor b, \Delta} \quad \lor R
\]

(If the rule holds in the whole universe, then it holds in the part where \( \land \Gamma \land \neg \lor \Delta \) holds.)
We’ve seen that contraposition generalizes to:

\[
\begin{align*}
\Gamma \models a, \Delta \quad \neg L \quad & \quad \Gamma, a \models \Delta \quad \neg R \\
\Gamma, \neg a \models \Delta \quad & \quad \Gamma \models \neg a, \Delta
\end{align*}
\]

We can take any formula, negate it and change which side it’s on.
There is one other rather obvious rule we need:

$$\Gamma, a \models a, \Delta$$

(You can think of this as the base case that finishes off the long recursive call that is a proof.)
Often we want to prove that some formula $a$ is universally valid or a tautology – valid in every universe.

This amounts to proving

$$\vdash a$$

‘$a$ is true with no assumptions’.

For example:

$$(¬p ∨ q) ∧ ¬p) ∨ p$$

is a tautology (think about it...).

We prove this by building a proof tree using the rules.
A proof!

\[
\Gamma, a \models (¬p ∨ q) ∧ ¬p) ∨ p
\]
A proof!

\[ \Gamma, a \vdash a, \Delta \]
\[ \Gamma \vdash a, \Delta \quad \neg L \]
\[ \Gamma, \neg a \vdash \Delta \]
\[ \Gamma, a \vdash \Delta \quad \neg R \]
\[ \Gamma, a, b \vdash \Delta \quad \land L \]
\[ \Gamma, a \land b \vdash \Delta \]
\[ \Gamma \vdash a, \Delta \quad \Gamma \vdash b, \Delta \quad \land R \]
\[ \Gamma \vdash a \land b, \Delta \]
\[ \Gamma, a \vdash \Delta \quad \Gamma, b \vdash \Delta \quad \lor L \]
\[ \Gamma, a \lor b \vdash \Delta \]
\[ \Gamma, a \lor b \vdash \Delta \quad \lor R \]
\[ \Gamma \vdash a \lor b, \Delta \]
A proof!

\[\Gamma, \neg a \vdash \Delta\]
\[\Gamma \vdash a, \Delta\]
\[\neg L\]
\[\Gamma, \neg a \vdash \Delta\]
\[\Gamma, a \vdash \Delta\]
\[\neg R\]
\[\Gamma, a \vdash \Delta\]
\[\neg L\]
\[\Gamma, a \vdash \Delta\]
\[\Gamma, b \vdash \Delta\]
\[\wedge R\]
\[\Gamma, a \vdash \Delta\]
\[\Gamma, b \vdash \Delta\]
\[\vee R\]
\[\Gamma, a \vdash \Delta\]
\[\Gamma, b \vdash \Delta\]
\[\vee L\]
\[\Gamma, a \vdash \Delta\]
\[\Gamma, b \vdash \Delta\]
\[\vee R\]
A proof!

\[
\begin{array}{c}
\Gamma, a \models a, \Delta \\
\Gamma \models a, \Delta \\
\Gamma, \neg a \models \Delta \\
\Gamma, a \models \Delta \\
\Gamma, a, b \models \Delta \\
\Gamma, a \land b \models \Delta \\
\Gamma \models a, \Delta \\
\Gamma \models b, \Delta \\
\Gamma \models a \land b, \Delta \\
\Gamma \models a, \Delta \\
\Gamma \models b, \Delta \\
\Gamma \models a \lor b, \Delta \\
\Gamma \models a \lor b, \Delta \\
\end{array}
\]
A proof!

\[ \Gamma, a \models a, \Delta \]
\[ \Gamma, a \models a, \Delta \]
\[ \Gamma, \neg a \models \Delta \]
\[ \Gamma, a \models \Delta \]
\[ \Gamma, a, b \models \Delta \]
\[ \Gamma, a \models \Delta \]
\[ \Gamma, b \models \Delta \]
\[ \Gamma, a \land b \models \Delta \]
\[ \Gamma, a \lor b \models \Delta \]
A proof!

\[
\begin{align*}
\Gamma, a &\models a, \Delta \\
\Gamma &\models a, \Delta \\
\Gamma, \neg a &\models \Delta \\
\Gamma, a &\models \Delta \\
\Gamma, a, b &\models \Delta \\
\Gamma &\models a \land b, \Delta \\
\Gamma, a &\models \Delta \\
\Gamma, b &\models \Delta \\
\Gamma &\models a \lor b, \Delta \\
\Gamma, a, \neg a &\models \Delta \\
\Gamma, a &\models \Delta \\
\Gamma, b &\models \Delta \\
\Gamma &\models a \land b, \Delta \\
\Gamma, a &\models \Delta \\
\Gamma, b &\models \Delta \\
\Gamma &\models a \lor b, \Delta \\
\Gamma, a &\models p \\
\Gamma &\models p \\
\Gamma, a &\models \Delta \\
\Gamma, b &\models \Delta \\
\Gamma &\models a \lor b, \Delta \\
\Gamma &\models \neg p \land q, \neg p \\
\Gamma &\models \neg p \lor q, \neg p \\
\Gamma &\models (\neg p \lor q) \land \neg p, \Delta \\
\Gamma &\models ((\neg p \lor q) \land \neg p) \lor p
\end{align*}
\]
A proof!

\[
\begin{align*}
&\frac{p \models q, p}{\models \neg p, q, p} \quad \neg R \\
&\frac{\models \neg p, q, p}{\models \neg p \lor q, p} \quad \lor R \\
&\frac{\models \neg p \lor q, p}{\models ((\neg p \lor q) \land \neg p), p} \quad \land R \\
&\frac{\models ((\neg p \lor q) \land \neg p) \lor p}{r} \\
\end{align*}
\]

\[
\begin{align*}
&\frac{\Gamma, a \models a, \Delta}{\Gamma \models a, \Delta} \quad \lor L \\
&\frac{\Gamma \models a, \Delta}{\Gamma, \neg a \models \Delta} \quad \neg L \\
&\frac{\Gamma, a \models a, \Delta}{\Gamma \models \neg a, \Delta} \quad \neg R \\
&\frac{\Gamma, a, b \models \Delta}{\Gamma, a \land b \models \Delta} \quad \land L \\
&\frac{\Gamma \models a, \Delta \quad \Gamma \models b, \Delta}{\Gamma \models a \land b, \Delta} \quad \land R \\
&\frac{\Gamma, a \models a \land b, \Delta}{\Gamma \models a \lor b, \Delta} \quad \lor L \\
&\frac{\Gamma, a \models a \land b, \Delta}{\Gamma, a \lor b \models \Delta} \quad \lor R \\
&\frac{\Gamma, a, b \models \Delta}{\Gamma \models a \lor b, \Delta} \quad \lor R
\end{align*}
\]

A proof!

\[
\begin{align*}
&\vdash \neg p, q, p \\
&\vdash \neg p \lor q, p \\
&\vdash ((\neg p \lor q) \land \neg p), p \\
&\vdash ((\neg p \lor q) \land \neg p) \lor p
\end{align*}
\]

\[
\begin{align*}
&\Gamma, a \vdash a, \Delta \\
&\Gamma \vdash a, \Delta \\
&\Gamma, \neg a \vdash \Delta \\
&\Gamma, a \vdash \Delta \\
&\Gamma, \neg a, \Delta \\
&\Gamma, a, b \vdash \Delta \\
&\Gamma \vdash a \land b, \Delta \\
&\Gamma, a \vdash \Delta \\
&\Gamma, b \vdash \Delta \\
&\Gamma \vdash a \lor b, \Delta \\
&\Gamma, a \lor b \vdash \Delta \\
&\Gamma, a, \Delta \\
&\Gamma, b, \Delta \\
&\Gamma \vdash a \lor b, \Delta
\end{align*}
\]
A proof!

\[\begin{array}{c}
\Gamma, a \vDash a, \Delta \\
\Gamma \vDash a, \Delta \\
\Gamma, \neg a \vDash \Delta \\
\Gamma, a \vDash \Delta \\
\Gamma, a, b \vDash \Delta \\
\Gamma \vDash a \wedge b, \Delta \\
\Gamma, a \vDash \Delta \\
\Gamma, b \vDash \Delta \\
\Gamma \vDash a \vee b, \Delta
\end{array}\]

So we have proved the formula with no assumptions.
And this was purely mechanical – we never had to think!
Finding necessary assumptions

Is $\neg((\neg a \lor b) \land (\neg c \lor b)) \lor (\neg a \lor c)$ a tautology?

\[
\begin{array}{c}
\Gamma, a \nvdash a, \Delta \\
\Gamma \nvdash a, \Delta \\
\Gamma, \neg a \nvdash \Delta \\
\Gamma, a \nvdash \Delta \\
\Gamma \nvdash \neg a, \Delta \\
\Gamma, a, b \nvdash \Delta \\
\Gamma, a \land b \nvdash \Delta \\
\Gamma \nvdash a, \Delta \\
\Gamma \nvdash b, \Delta \\
\Gamma \nvdash a \land b, \Delta \\
\Gamma, a \nvdash \Delta \\
\Gamma, b \nvdash \Delta \\
\Gamma \nvdash a \lor b, \Delta \\
\Gamma \nvdash a, b, \Delta \\
\Gamma \nvdash a \lor b, \Delta \\
\end{array}
\]
Finding necessary assumptions

Is $\neg((\neg a \lor b) \land (\neg c \lor b)) \lor (\neg a \lor c)$ a tautology?

Try to prove it:

$$\vdash \neg((\neg a \lor b) \land (\neg c \lor b)) \lor (\neg a \lor c)$$
Finding necessary assumptions

Is \( \neg((\neg a \lor b) \land (\neg c \lor b)) \lor (\neg a \lor c) \) a tautology?

Try to prove it:

\[
\begin{align*}
\Gamma, a \vdash a, \Delta \\
\Gamma, \neg a \vdash \Delta \\
\Gamma, a \vdash \Delta \\
\Gamma, \neg a \vdash \Delta \\
\Gamma, a, b \vdash \Delta \\
\Gamma, a \land b \vdash \Delta \\
\Gamma, a, b \vdash \Delta \\
\Gamma, b \vdash \Delta \\
\Gamma, a \land b \vdash \Delta \\
\Gamma, a \lor b \vdash \Delta \\
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash \neg((\neg a \lor b) \land (\neg c \lor b)), \quad (\neg a \lor c) \\
\vdash \neg((\neg a \lor b) \land (\neg c \lor b)) \lor (\neg a \lor c)
\end{align*}
\]
Finding necessary assumptions

Is \( \neg((\neg a \lor b) \land (\neg c \lor b)) \lor (\neg a \lor c) \) a tautology?

Try to prove it:

\[
\begin{align*}
(\neg a \lor b) \land (\neg c \lor b) & \vdash \neg a \lor c \\
\vdash \neg((\neg a \lor b) \land (\neg c \lor b)), \ (\neg a \lor c) \\
\vdash \neg((\neg a \lor b) \land (\neg c \lor b)) \lor (\neg a \lor c)
\end{align*}
\]
Is \( \neg((\neg a \lor b) \land (\neg c \lor b)) \lor (\neg a \lor c) \) a tautology?

Try to prove it:

\[
\begin{align*}
\neg a \lor b, \neg c \lor b \models & \neg a, c & \land, \lor \ R \\
(\neg a \lor b) \land (\neg c \lor b) \models & \neg a \lor c & \neg R \\
\models & \neg((\neg a \lor b) \land (\neg c \lor b)), (\neg a \lor c) & \lor \ R \\
\models & \neg((\neg a \lor b) \land (\neg c \lor b)) \lor (\neg a \lor c)
\end{align*}
\]
Finding necessary assumptions

Is \( \neg((\neg a \lor b) \land (\neg c \lor b)) \lor (\neg a \lor c) \) a tautology?
Try to prove it:

\[
\begin{align*}
\neg a, \neg c \lor b \models \neg a, c & \quad b, \neg c \lor b \models \neg a, c \\
\neg a \lor b, \neg c \lor b \models \neg a \lor c & \quad \lor L \\
(\neg a \lor b) \land (\neg c \lor b) \models \neg a \lor c & \quad \land L, \lor R \\
\models \neg((\neg a \lor b) \land (\neg c \lor b)), (\neg a \lor c) & \quad \neg R \\
\models \neg((\neg a \lor b) \land (\neg c \lor b)) \lor (\neg a \lor c) & \quad \lor R
\end{align*}
\]
Finding necessary assumptions

Is \( \neg((\neg a \lor b) \land (\neg c \lor b)) \lor (\neg a \lor c) \) a tautology?

Try to prove it:

\[
\begin{align*}
\neg a, \neg c \lor b & \vdash \neg a, c & b, b \vdash \neg a, c & \lor L \\
\neg a \lor b, \neg c \lor b & \vdash \neg a, c & b, \neg c \lor b & \vdash \neg a, c & \lor L \\
(\neg a \lor b) \land (\neg c \lor b) & \vdash \neg a \lor c & \lor R \\
\vdash \neg((\neg a \lor b) \land (\neg c \lor b)), (\neg a \lor c) & \lor R \\
\vdash \neg((\neg a \lor b) \land (\neg c \lor b)) \lor (\neg a \lor c) & \\
\end{align*}
\]
Finding necessary assumptions

Is \( \neg((\neg a \lor b) \land (\neg c \lor b)) \lor (\neg a \lor c) \) a tautology?

Try to prove it:

\[
\begin{align*}
\Gamma, \neg a, \Delta & \vdash \neg a, c \\
\therefore b, \neg c & \vdash \neg a, c \\
\therefore b, \neg c \lor b & \vdash \neg a, c \\
\therefore \neg a \lor b, \neg c \lor b & \vdash \neg a, c \\
\therefore \neg a \lor c & \vdash \neg a \lor c \\
\therefore \neg((\neg a \lor b) \land (\neg c \lor b)), \neg a \lor c & \vdash \neg((\neg a \lor b) \land (\neg c \lor b)) \lor (\neg a \lor c)
\end{align*}
\]
Is \( \neg((\neg a \lor b) \land (\neg c \lor b)) \lor (\neg a \lor c) \) a tautology?

Try to prove it:

\[
\begin{align*}
\neg a, \neg c \lor b & \models \neg a, c \\
\because \quad b, \neg c & \models \neg a, c \\
\therefore \quad a, b & \models c \\
\therefore \quad \neg R, \neg L \\
\because \quad a, b & \models c \\
\therefore \quad b, \neg c & \models \neg a, c \\
\therefore \quad \neg R \\
\therefore \quad \lor L \\
\therefore \quad b, \neg c \lor b & \models \neg a, c \\
\therefore \quad \lor L \\
\therefore \quad \neg a \lor b, \neg c \lor b & \models \neg a \lor c \\
\therefore \quad \land L, \lor R \\
\therefore \quad \neg a \lor b, \neg c \lor b & \models \neg a \lor c \\
\therefore \quad \lor R \\
\therefore \quad \neg((\neg a \lor b) \land (\neg c \lor b)) & \models \neg a \lor c \\
\therefore \quad \lor R \\
\therefore \quad \neg((\neg a \lor b) \land (\neg c \lor b)) \lor (\neg a \lor c) & \models \neg R \\
\therefore \quad \lor R \\
\therefore \quad \neg((\neg a \lor b) \land (\neg c \lor b)) & \models (\neg a \lor c) \\
\therefore \quad \lor R \\
\therefore \quad \neg((\neg a \lor b) \land (\neg c \lor b)) \lor (\neg a \lor c) & \models \lor L \\
\therefore \quad \lor R \\
\therefore \quad \neg((\neg a \lor b) \land (\neg c \lor b)) \lor (\neg a \lor c) & \models \lor R
\end{align*}
\]
Is \( \neg((\neg a \lor b) \land (\neg c \lor b)) \lor (\neg a \lor c) \) a tautology?

Try to prove it:

\[
\frac{a, b \vdash c}{\neg R, \neg L} \quad \frac{a, b \vdash c}{\neg R} \quad \frac{b, b \vdash \neg a, c}{\lor L}
\]

\[
\frac{b, \neg c \vdash \neg a, c}{\lor R}
\]

\[
\frac{a \lor b, \neg c \lor b \vdash \neg a, c}{\land, \lor R}
\]

\[
\frac{\neg a \lor b, \neg c \lor b \vdash \neg a \lor c}{\land, \lor R}
\]

\[
\frac{\neg((\neg a \lor b) \land (\neg c \lor b)) \lor (\neg a \lor c)}{\lor R}
\]

\[
\frac{\neg((\neg a \lor b) \land (\neg c \lor b)) \lor (\neg a \lor c)}{\lor R}
\]
Is \( \neg((\neg a \lor b) \land (\neg c \lor b)) \lor (\neg a \lor c) \) a tautology?

Try to prove it:

We are left with needing to assume \( a, b \models c \) (or equivalently, \( \models \neg a, \neg b, c \)).
Assumptions provide counter-examples

The last slide showed (since all the rules work backwards) that

\[
a, b \models c
\]

\[
\neg((\neg a \lor b) \land (\neg c \lor b)) \lor (\neg a \lor c)
\]

So a counter-example to the conclusion is any universe where \( a, b \models c \) fails: i.e. one where something is \( a \) and \( b \) but not \( c \).

For example, the universe of things from week 1, taking \( a \) as ‘small’, \( b \) as ‘triangle’, and \( c \) as ‘red’.
If you try to prove a tautology, you succeed, and every leaf of the proof tree is an $I$-rule.

If you try to prove a non-tautology, you get leaves with assumptions. Denying any of those assumptions gives a counter-example.

What happens when you try to prove a universally false statement? Try with

$$\models a \land \neg a$$