Informatics 1 – Introduction to Computation
Computation and Logic

Julian Bradfield
based on materials by
Michael P. Fourman

Satisfying Assignments
Boolean Algebra, Tseytin, Counting

Henry Scheffer, 1882–1964

Gregory Tseytin, 1936–2022

Boolean operators – recap

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name	sym	t.t.	a.k.a.
true	$\top$	1	1, top
false	$\bot$	0	0, bottom
not	$\neg$	1 0	complement, $\bar{\ }$
and	$\wedge$	0 0 0 1	&, ., $\times$
or	$\vee$	0 1 1 1	|, +
implies	$\to$	1 1 0 1	$\leq$

name	sym	t.t.	a.k.a.
implied by	$\leftarrow$	1 0 1 1	$\geq$
iff	$\leftrightarrow$	1 0 0 1	$=$
xor	$\oplus$	0 1 1 0	$\neq$, parity
nand	$\overline{\wedge}$	1 1 1 0	
nor	$\overline{\vee}$	1 0 0 0	

Using boolean operators

Everything we've done with boolean operators can be extended to use $\to$, $\leftrightarrow$ and others.

In the optional question of tutorial 5, you were asked for sequent calculus rules for $\to$. They are:

$$\frac{\Gamma, b \vDash \Delta \quad \Gamma \vDash a, \Delta}{\Gamma, a \to b \vDash \Delta} \; (\to L) \qquad \frac{\Gamma, a \vDash b, \Delta}{\Gamma \vDash a \to b, \Delta} \; (\to R)$$

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Rules for $\leftrightarrow$ are even more obvious:

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Boring exercise: take all the stuff you've done in Haskell on WFFs etc., and extend it for these operators, if you haven't already.

Note that $(\rightarrow R)$ has the special case

$$\begin{array}{c} a \models b \\ \hline\hline \models a \rightarrow b \end{array}$$

which ties down the precise similarity between $\models$ and $\rightarrow$.

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This set of axioms is far from minimal. Astonishingly, this single axiom suffices: $\neg(\neg(\neg(a \vee b) \vee c) \vee \neg(a \vee \neg(\neg c \vee \neg(c \vee d)))) = c$
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Other convenient derived equations include:

- Negation cancellation: $\neg\neg a = a$
- Zero/One: $\neg 1 = 0$ and $\neg 0 = 1$
- Simple absorption: $a \vee a = a$ and sim. for $\wedge$
- De Morgan: $\neg(a \vee b) = \neg a \wedge \neg b$ and *vice versa*

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Proof $\neg y = 1 \wedge \neg y$ by identity

$= (x \vee y) \wedge \neg y$ by assumption

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By the lemma, to prove $\neg(a \vee b) = \neg a \wedge \neg b$, it suffices to prove $(a \vee b) \vee (\neg a \wedge \neg b) = 1$ and $(a \vee b) \wedge (\neg a \wedge \neg b) = 0$. Both these follow easily by distributivity, complement, associativy and commutativity: e.g. the first is
$(a \vee b) \vee (\neg a \wedge \neg b) = ((a \vee b) \vee \neg a) \wedge ((a \vee b) \vee \neg b) = \cdots = 1 \wedge 1 = 1$.

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We can add derived rules, as we did in sequent calculus:

- Bi-implication: $a \leftrightarrow b = (a \rightarrow b) \land (b \rightarrow a)$
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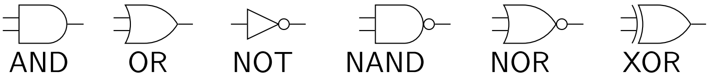
Progamming this was in FP tutorial 6! If you didn't try the optional and challenge parts, go back and try them now.

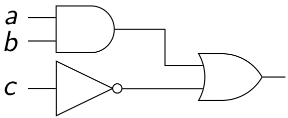
Doing this by hand tends to be boring: see textbook chapter 22 for worked examples.

(Not a) Short Digression: Circuits

Ultimately, logic is implemented in silicon via transistors, referred to as logic gates. Circuit designers draw gates like this:



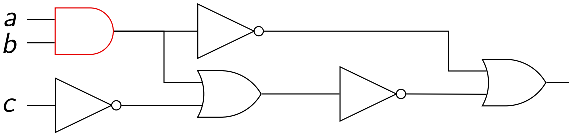
Gates (boolean operators) are connected by drawing wires:



is the circuit for $(a \wedge b) \vee \neg c$.

Circuits can duplicate expressions

A circuit can use the same output more than once:



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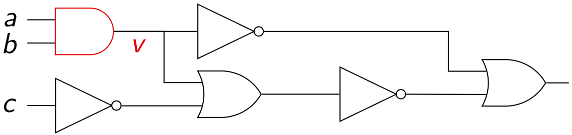

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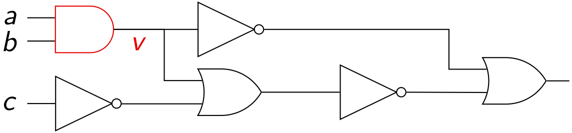

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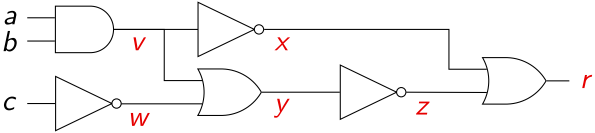

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We can do this for all the intermediate values, and forget the original formula.

Note that we include the variable for the whole formula: this variable needs to be true.

r
$r \leftrightarrow x \vee z$
$x \leftrightarrow \neg v$
$v \leftrightarrow a \wedge b$
$z \leftrightarrow \neg y$
$y \leftrightarrow v \vee w$
$w \leftrightarrow \neg c$

The Tseytin transformation

does with formulae what we've just done with gates.

Introduce a new variable x for every subformula ϕ, and add a clause saying $x \leftrightarrow \phi$. For example:

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Tseytin is an $O(n)$ conversion to an equisatisfiable CNF formula.

Unfortunately CNF-SAT can still be exponential – no free lunch.

Final question for you: how long does it take to check satisfiability of a DNF formula?

2-CNF-SAT

2-CNF-SAT (or just 2-SAT) is the special case where *every clause has at most two literals*, such as:

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They arise naturally in problems involving may/must/must not relations between things: e.g. which courses you are able to take. Sometimes unfortunate consequences arise from simple rules . . .

2-variable clauses

Any two-variable clause can be written in terms of $\vee$ and $\neg$, and vice versa.

Rewriting the previous out of CNF gives:

$$\neg(A \wedge C) \wedge (B \to C) \wedge (A \vee B) \wedge (C \to D) \wedge \neg(D \wedge B)$$

which might represent the following rules:

1. You may not take both Astrology and Chiromancy
2. If you take Belomancy, you must take Chiromancy
3. You must take Astrology or Belomancy
4. If you take Chiromancy, you must take Dream Interpretation
5. You may not take both Dream Interpretation and Belomancy

What can you take?

Implication clauses

Any two-variable clause can *also* be written in terms of $\rightarrow$ and $\neg$:

$$(A \rightarrow \neg C) \wedge (B \rightarrow C) \wedge (\neg A \rightarrow B) \wedge (C \rightarrow D) \wedge (D \rightarrow \neg B)$$

$\neg A \vee \neg C$ is symmetrical. Is $A \rightarrow \neg C$ symmetrical? (Remember back to sequents and contraposition. . .)

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This is useful because *implication is transitive*:
if $B \rightarrow C$ and $C \rightarrow D$, then $B \rightarrow D$

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Remember *barbara*!

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$$(A \rightarrow \neg C) \wedge (B \rightarrow C) \wedge (\neg A \rightarrow B) \wedge (C \rightarrow D) \wedge (D \rightarrow \neg B)$$

This is useful because *implication is transitive*:
if $B \rightarrow C$ and $C \rightarrow D$, then $B \rightarrow D$

We can build a partial **graph** of implication between literals:

$$0 \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow \neg B \longrightarrow 1$$

$\neg A \vee \neg C$ is symmetrical. Is $A \rightarrow \neg C$ symmetrical? (Remember back to sequents and contraposition. . .)

Remember *barbara*!

$0 \rightarrow$ anything, and anything $\rightarrow 1$

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Satisfying assignments are got from cutting the line somewhere, which must be right of B. (And then dealing with the rest.)

$\neg A \vee \neg C$ is symmetrical. Is $A \rightarrow \neg C$ symmetrical? (Remember back to sequents and contraposition...)

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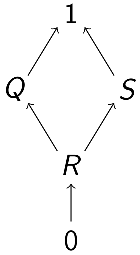
The Arrow Rule

says that if we draw the full graph of implications, any valid cut through the graph gives a satisfying assignment: literals above the cut are true, those below are false. Another example:

$$(\neg R \lor Q) \land (\neg R \lor S) \quad \text{equiv} \quad (R \to Q) \land (R \to S)$$



A cut is a set of edges which, when deleted, cut the graph in two.

A valid cut must separate 0 and 1.

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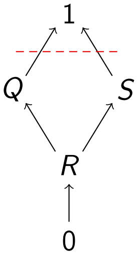

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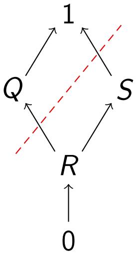

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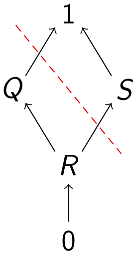

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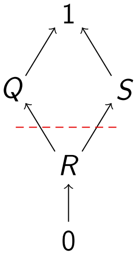

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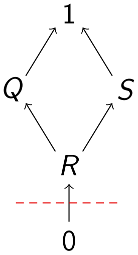

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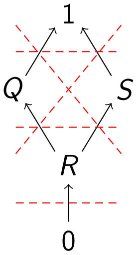


There are five satisfying assignments, one for each valid cut.

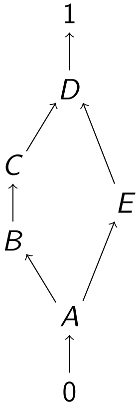
A cut is a set of edges which, when deleted, cut the graph in two.

A valid cut must separate 0 and 1.

Counting assignments

A more complex example:

$$(A \to B) \wedge (B \to C) \wedge (C \to D) \wedge (A \to E) \wedge (E \to D)$$



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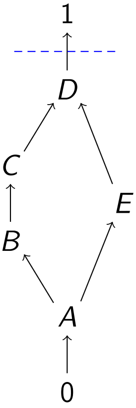

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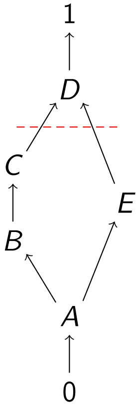

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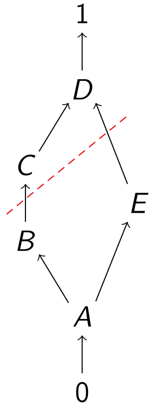

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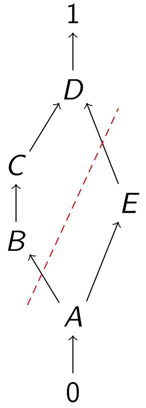

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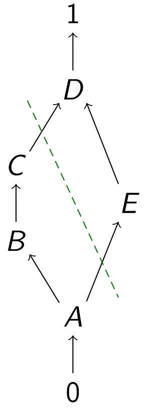

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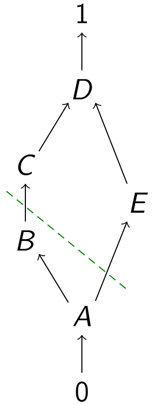

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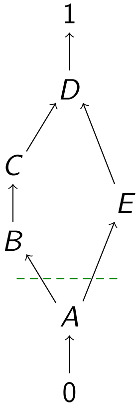

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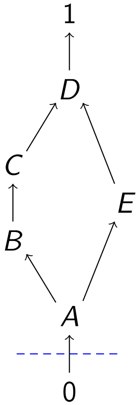
There are eight ways to cut this.



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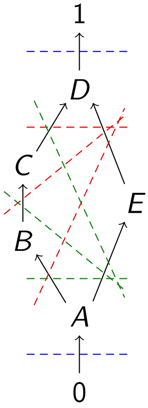
We can count cuts thus:

- one cut above D



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There are eight ways to cut this.

We can count cuts thus:

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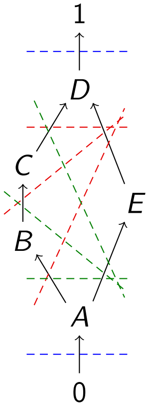

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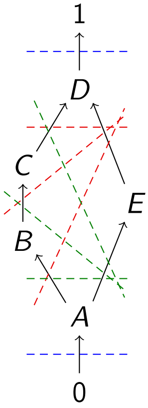

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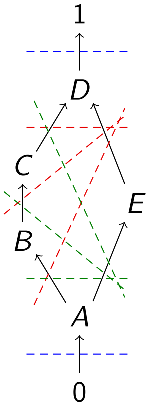
For an even more complicated example, see the textbook (Chapter 23, p. 252).



Complementary literals

What happens with formulae that have A and $\neg A$ (like the very first one)? Such as:

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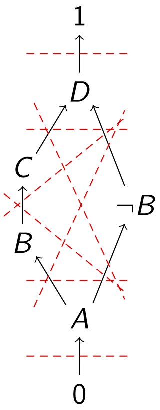

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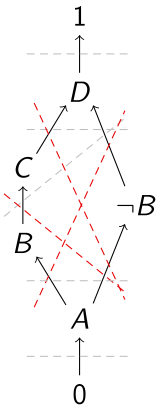

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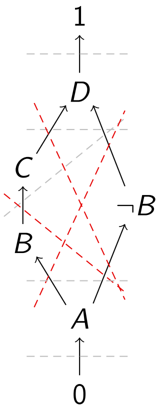
Note $A \to \neg B$ is the same as $B \to \neg A$ (contraposition), so sometimes you can remove complementary literals. This makes thing easier!



Cycles in the graph

It's quite possible for the implication graph to contain *cycles*. For example:

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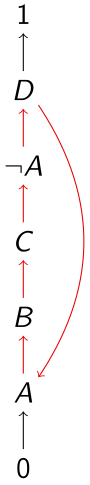

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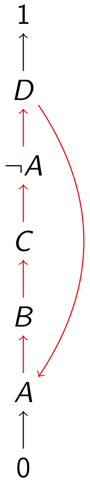

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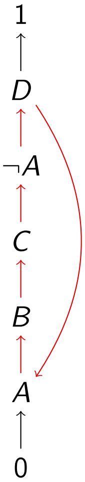

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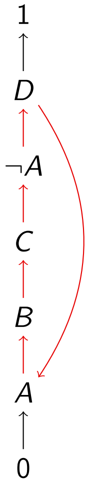

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Sometimes cycles can be removed by taking the contrapositive. Go back to the first example (slide 12) and complete it both with and without a cycle.

Summary

Drawing the implication graph and counting valid cuts lets us count satisfying assignments of 2-SAT formulae.

A valid cut must:

- separate 0 and 1
- separate complementary literals
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(There is one quirk we haven't considered. What if the implication graph is *non-planar*? See the book for how to deal with that.)