Inf2 - Foundations of Data Science: Estimation – Principle of confidence intervals

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Announcements

Project ideas - due Friday!
- Please contribute on the "Request for projects suggestions" Piazza post

Workshops - hopefully they've been helpful

Lab

Last Lecture

1. Parameter
   - value of a statistic (e.g. mean or max) in population
   - parameter in distribution (e.g. mean, variance of normal)

2. Point estimator
   - Method of converting sample into estimate of paramater
   - E.g. Mean of sample ($\bar{x}$) estimates mean of population μ

     $\hat{\mu}$

3. Point estimator is random variable
   - a different random sample from population =>
     different value of point estimator
   - But we only have one sample, so only one value

4. Idea of confidence intervals for estimator
   - based on sample standard error of estimator

Today

1. How to convert inferred sampling distribution of estimator into a confidence interval with a specified chance of enclosing true value

2. (a) How to compute a confidence interval for mean of large sample
   - z distribution

   (b) How to calculate a confidence interval for mean of a small sample from a fairly normal distribution
   - t distribution

3. Choosing confidence levels and how much data to collect

4. Confidence intervals of parameters other than the mean
   - Bootstrap

Inf2 - Foundations of Data Science:
Estimation –
Principle of confidence intervals
(exemplified by the mean of a large sample)

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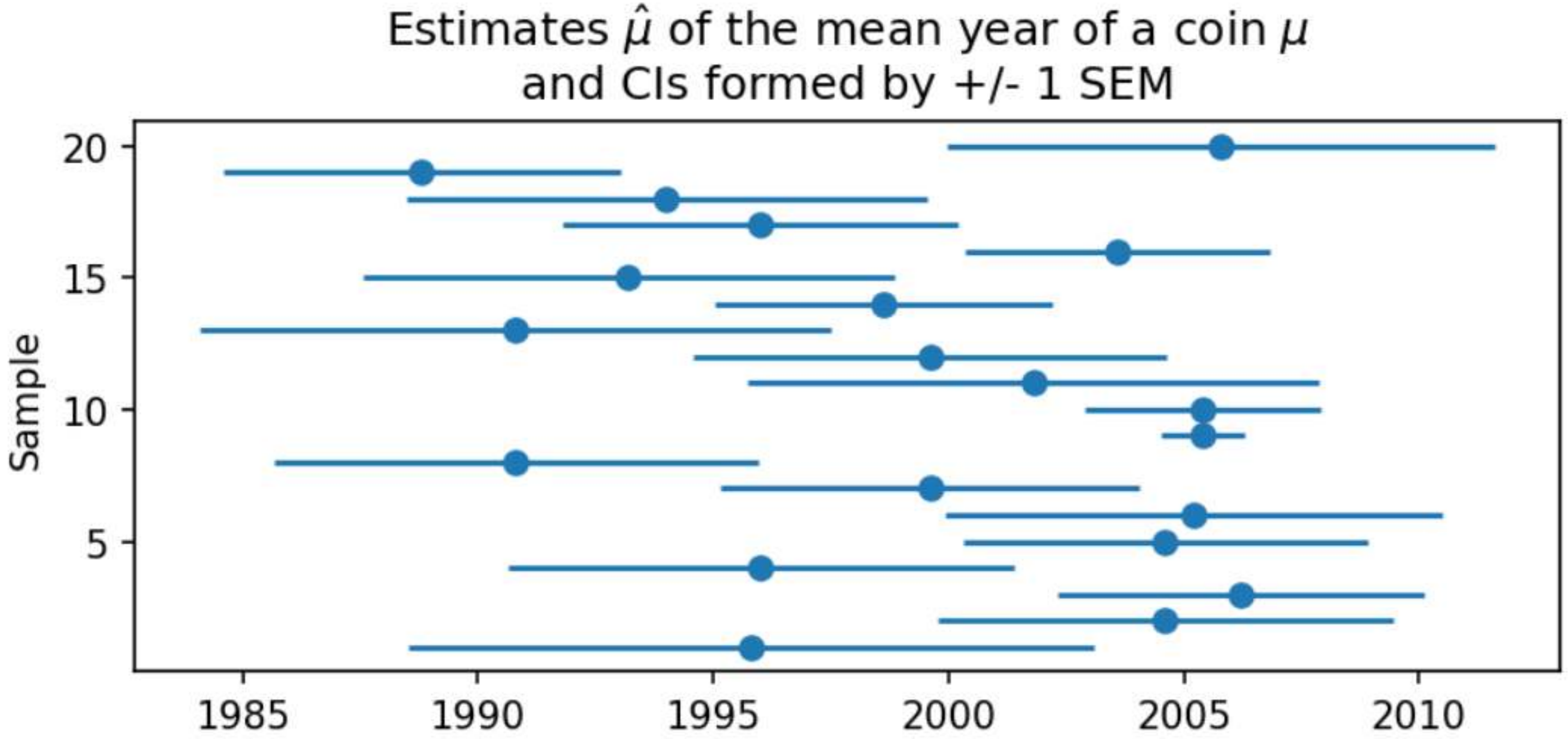
Confidence intervals from different samples



How do we calibrate the width of the confidence interval so that there is a specified chance that it encloses the true value?

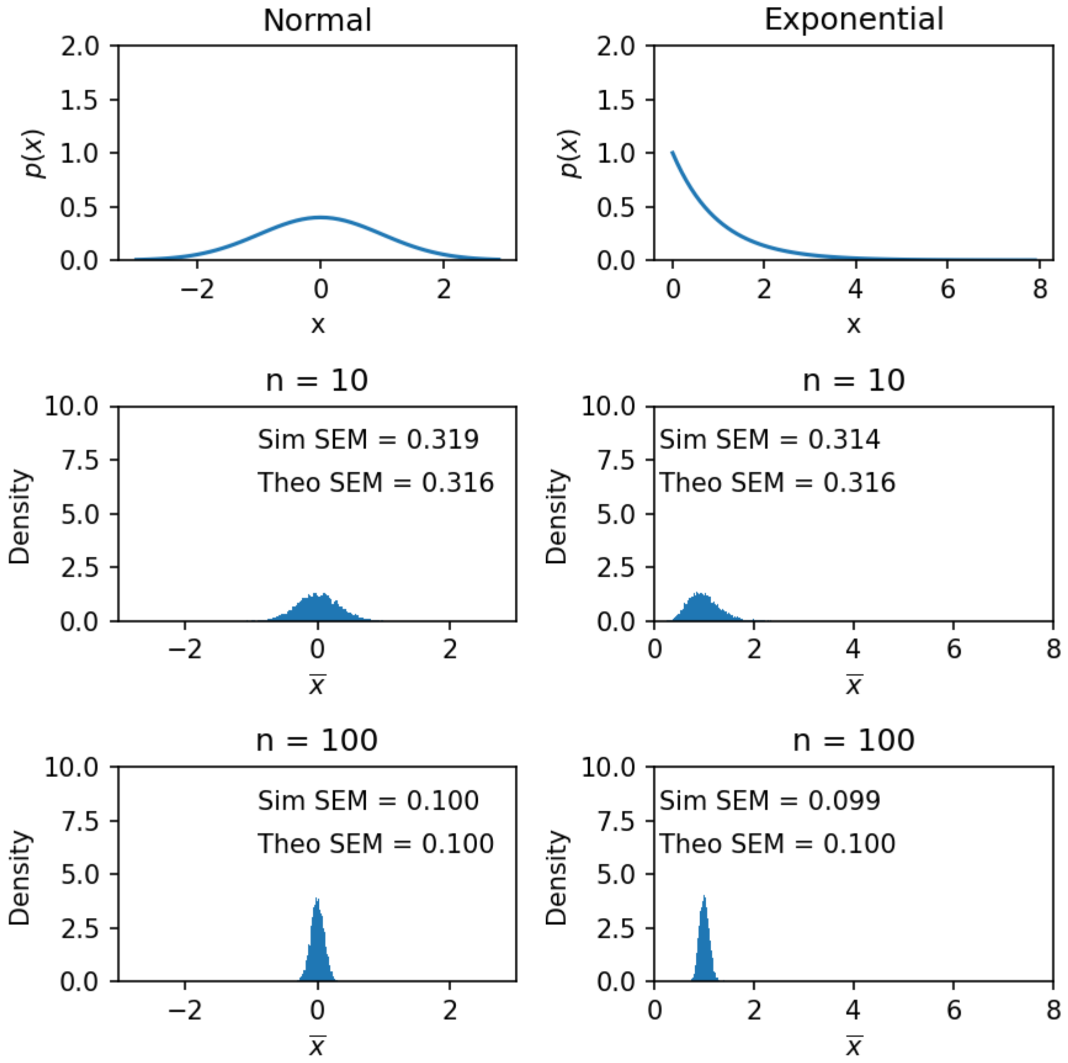
Theory reminder:
Standard error of the mean for known distribution variance σ

Standard deviation

$\sigma = 1$

SEM

$\frac{\sigma}{\sqrt{n}} = \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{10}} = 0.316$



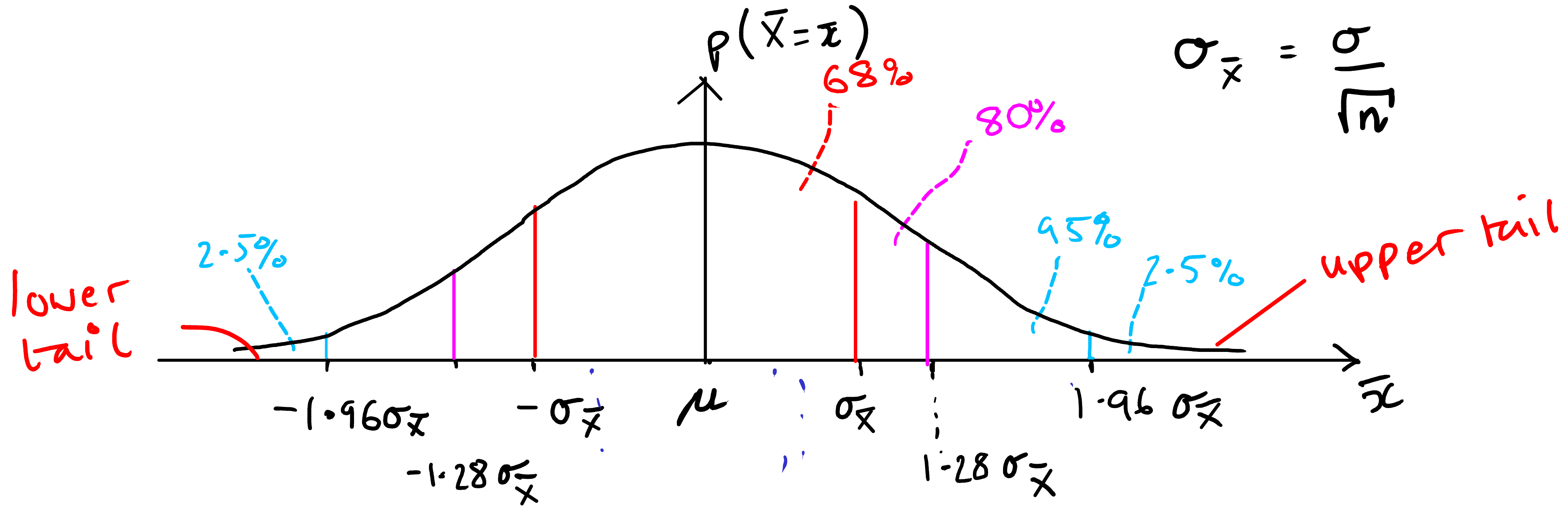
$\sigma_{\hat{\mu}} = \frac{1}{\sqrt{100}}$

CLT

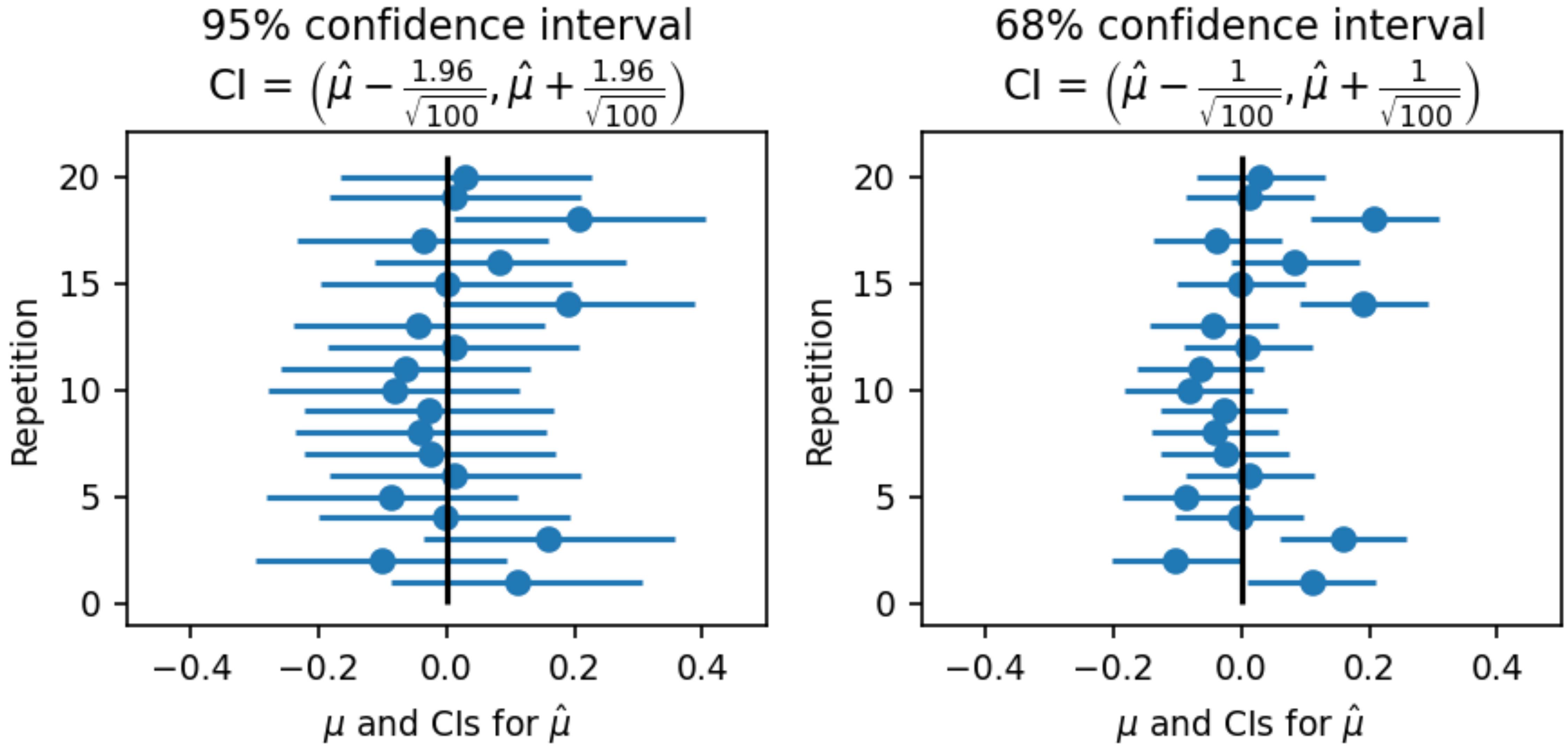
Normal

Approaching normal

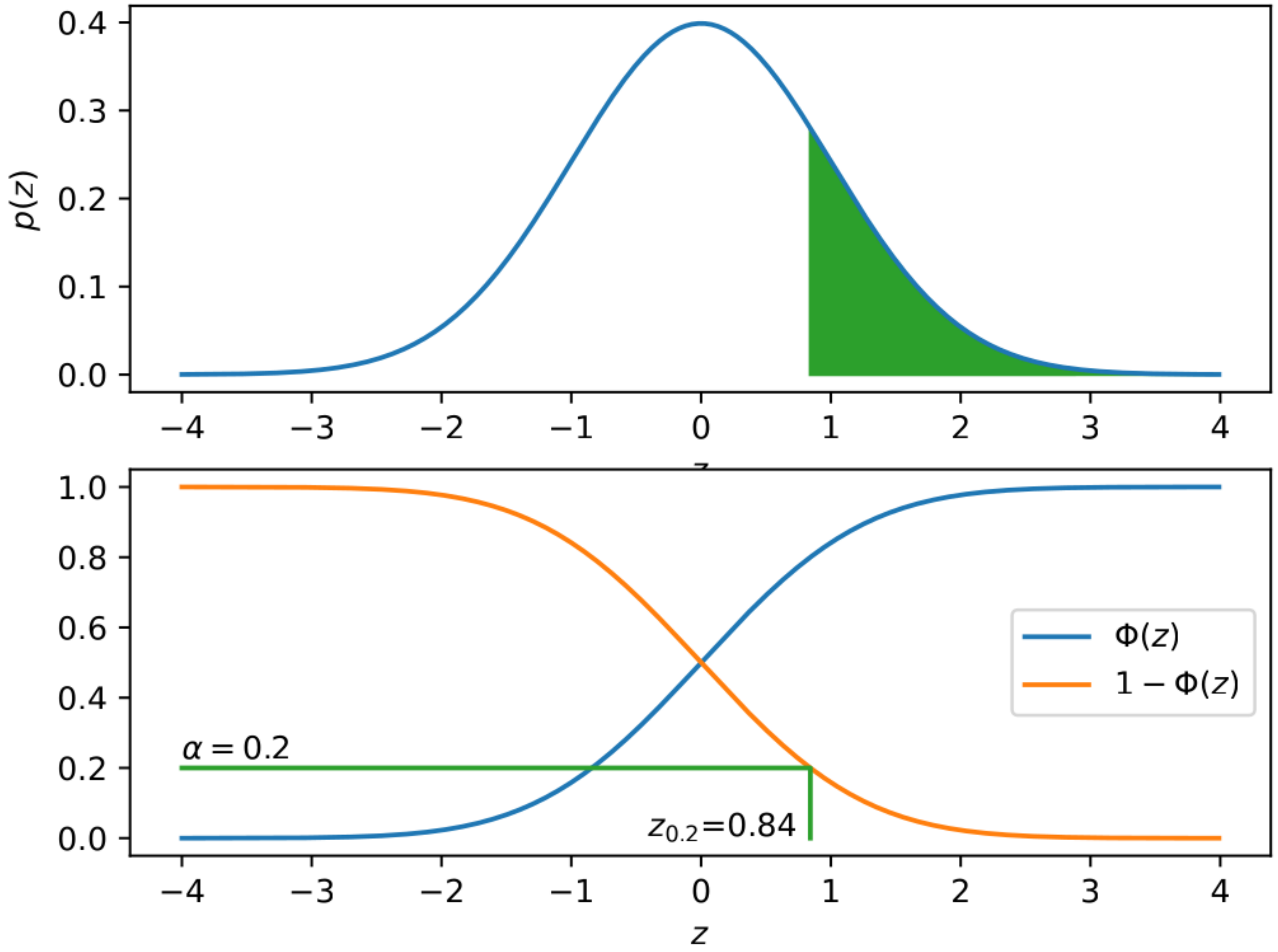
Confidence interval of the mean of a sample from a distribution with unknown mean and known variance



What we expect for confidence intervals of mean of 100 samples from normal distribution with mean 0 and variance 1



Determining the width of the interval: z-critical values



α	0.100	0.050	0.025	0.010	0.005	0.001
ν						
1	3.078	6.314	12.706	31.821	63.657	318.309
2	1.886	2.920	4.303	6.965	9.925	22.327
3	1.638	2.353	3.182	4.541	5.841	10.215
4	1.533	2.132	2.776	3.747	4.604	7.173
5	1.476	2.015	2.571	3.365	4.032	5.893
6	1.440	1.943	2.447	3.143	3.707	5.208
7	1.415	1.895	2.365	2.998	3.499	4.785
8	1.397	1.860	2.306	2.896	3.355	4.501
9	1.383	1.833	2.262	2.821	3.250	4.297
10	1.372	1.812	2.228	2.764	3.169	4.144
20	1.325	1.725	2.086	2.528	2.845	3.552
30	1.310	1.697	2.042	2.457	2.750	3.385
40	1.303	1.684	2.021	2.423	2.704	3.307
∞	1.282	1.645	1.960	2.326	2.576	3.090

$1-\alpha$ CI :

$$\left(\bar{x} - z_{\alpha/2}\frac{s}{\sqrt{n}}, \bar{x} + z_{\frac{\alpha}{2}}\frac{s}{\sqrt{n}}\right)$$

α	0.100	0.050	0.025	0.010	0.005	0.001
∞	1.282	1.645	1.960	2.326	2.576	3.090

Inf2 - Foundations of Data Science: Estimation – Definition of a confidence interval

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Definition of a confidence interval for a parameter

For an estimator $\hat{\vartheta}$ and estimated standard error of the estimator $\hat{\sigma}_{\hat{\vartheta}}$, the $100(1-\alpha)\%$ confidence interval

$$(\hat{\vartheta} - a\hat{\sigma}_{\hat{\vartheta}},\ \hat{\vartheta} + b\hat{\sigma}_{\hat{\vartheta}})$$

has a specified chance $1-\alpha$ of containing the parameter ϑ.

e.g. $\alpha = 0.05 \Rightarrow 1-0.05 = 95\%$ C.I.

$$P(\hat{\vartheta} - a\hat{\sigma}_{\hat{\vartheta}} < \vartheta < \hat{\vartheta} + b\hat{\sigma}_{\hat{\vartheta}}) = 1-\alpha$$

Often the interval is symmetric, ie $a = b$.

Rearranging the confidence interval definition so that there is one random variable

Definition :

$$P(\hat{\vartheta} - a\hat{\sigma}_{\hat{\vartheta}} < \vartheta < \hat{\vartheta} + b\hat{\sigma}_{\hat{\vartheta}}) = 1 - \alpha$$

$$\Rightarrow \quad P(a\hat{\sigma}_{\hat{\vartheta}} > \hat{\vartheta} - \vartheta > -b\hat{\sigma}_{\hat{\vartheta}}) = 1 - \alpha$$

$$\Rightarrow \quad P\left(a > \frac{\hat{\vartheta} - \vartheta}{\hat{\sigma}_{\hat{\vartheta}}} > -b\right) = 1 - \alpha$$

$$\Rightarrow \quad P\left(-b < \boxed{\frac{\hat{\vartheta} - \vartheta}{\hat{\sigma}_{\hat{\vartheta}}}} < a\right) = 1 - \alpha$$

standardised variable

random variable

How do we choose a and b for a particular estimator?

Consider the theoretical distribution of the standardised random variable $\dfrac{\hat{\theta} - \theta}{\hat{\sigma}_{\hat{\theta}}}$.

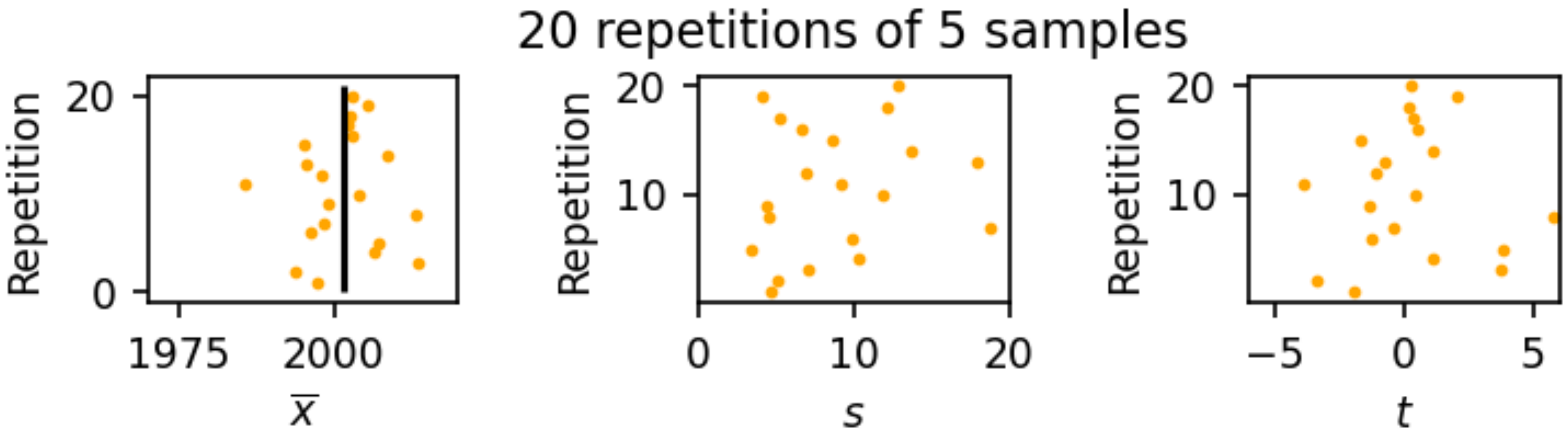
E.g. Estimator $\hat{\theta} = \hat{\mu} = \bar{X}$ of the mean μ

standard error in mean $\hat{\sigma}_{\hat{\mu}} = \dfrac{S}{\sqrt{n}}$

standardised r.v. $T = \dfrac{\bar{X} - \mu}{S/\sqrt{n}}$

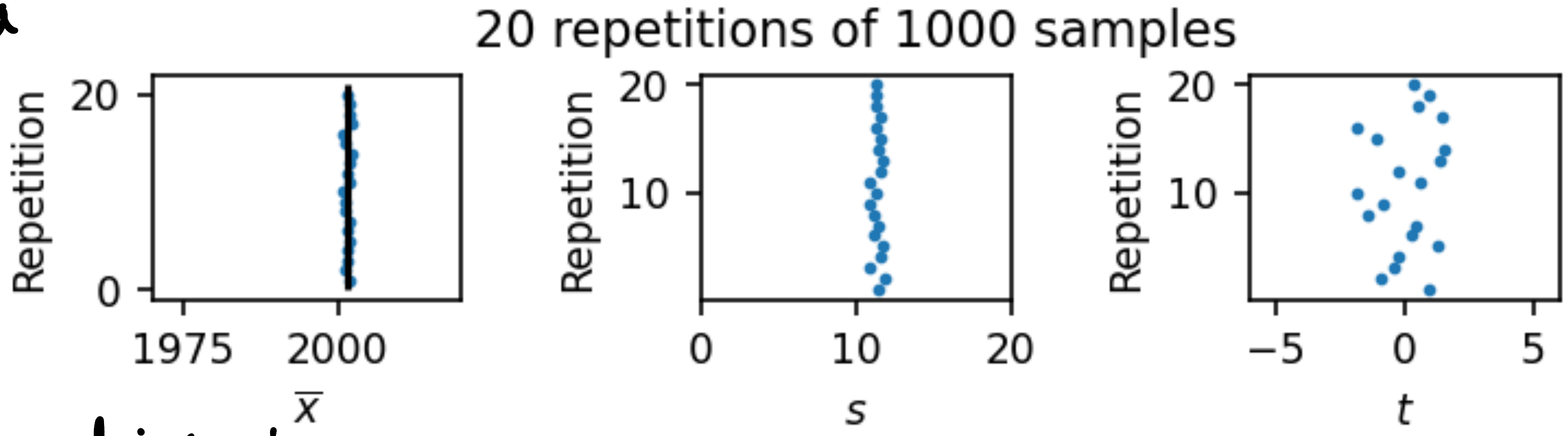
Inf2 - Foundations of Data Science: Estimation – Theoretical method of estimating the confidence interval of the mean of a large sample or a small, near-normal sample

Distribution of T



Where distribution of data is fairly normal

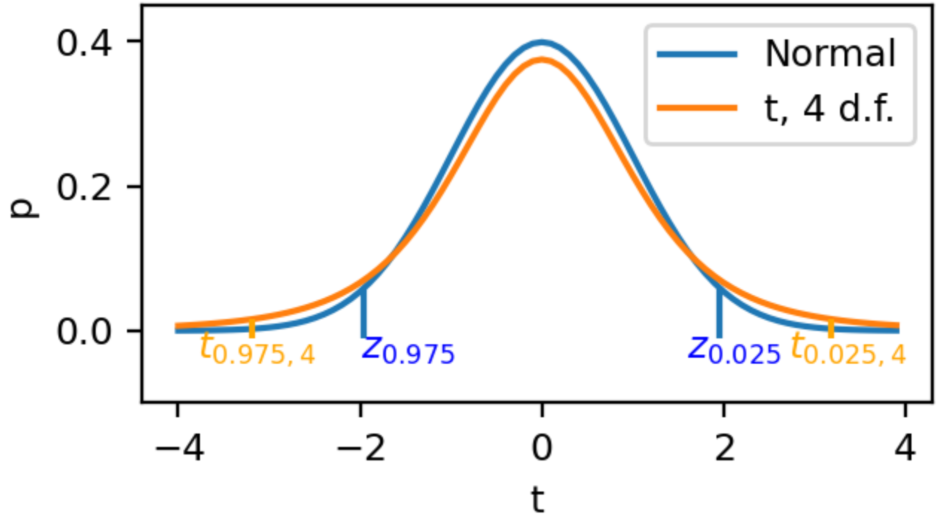
$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}}$$



distributed with a t-distribution with $\nu = n-1$ degrees of freedom

T-critical values $t_{\alpha,\nu}$

For $n > 40$ $\quad t_{\alpha,\nu} \simeq z_\alpha$



Small sample confidence interval example

$n = 29$ coins

$\bar{x} = 2001.551$ years $\qquad s = 11.444$ years

Estimated SEM, $\sigma_{\bar{x}} = \frac{s}{\sqrt{n}} = \frac{11.444}{\sqrt{29}} = 2.125$ years

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Assume distribution is fairly normal

$\Rightarrow \quad T = \frac{\bar{X} - \mu}{\hat{\sigma}_{\bar{x}}}$ is distributed according to a t-distribution with $n-1$ degrees of freedom.

Using the t-distribution to calculate a confidence interval

95% C.I $\Rightarrow$ $\alpha = 0.05$

Sample size $n \Rightarrow$ $\nu = n-1$ d.f.

$t_{\alpha/2, \nu} = t_{\frac{\alpha}{2}, n-1}$ t-critical value

$\bar{x} - t_{\frac{\alpha}{2}, n-1} \hat{\sigma}_{\bar{x}}$, $\bar{x} + t_{\frac{\alpha}{2}, n-1} \hat{\sigma}_{\bar{x}}$

$n = 29$, $\alpha = 0.05 \Rightarrow t_{0.025, 29-1} = t_{0.025, 28}$

$= 2.281$

$t_{0.025, 28} \hat{\sigma}_{\bar{x}} = 2.281 \times 2.125 = 4.871$ years

$\Rightarrow \hat{\mu} = 2001 \pm 5$ years (95% C.I.)

Estimated standard error of them mean for distribution with unknown variance σ

What if we don't know σ ?

Estimated S.E.

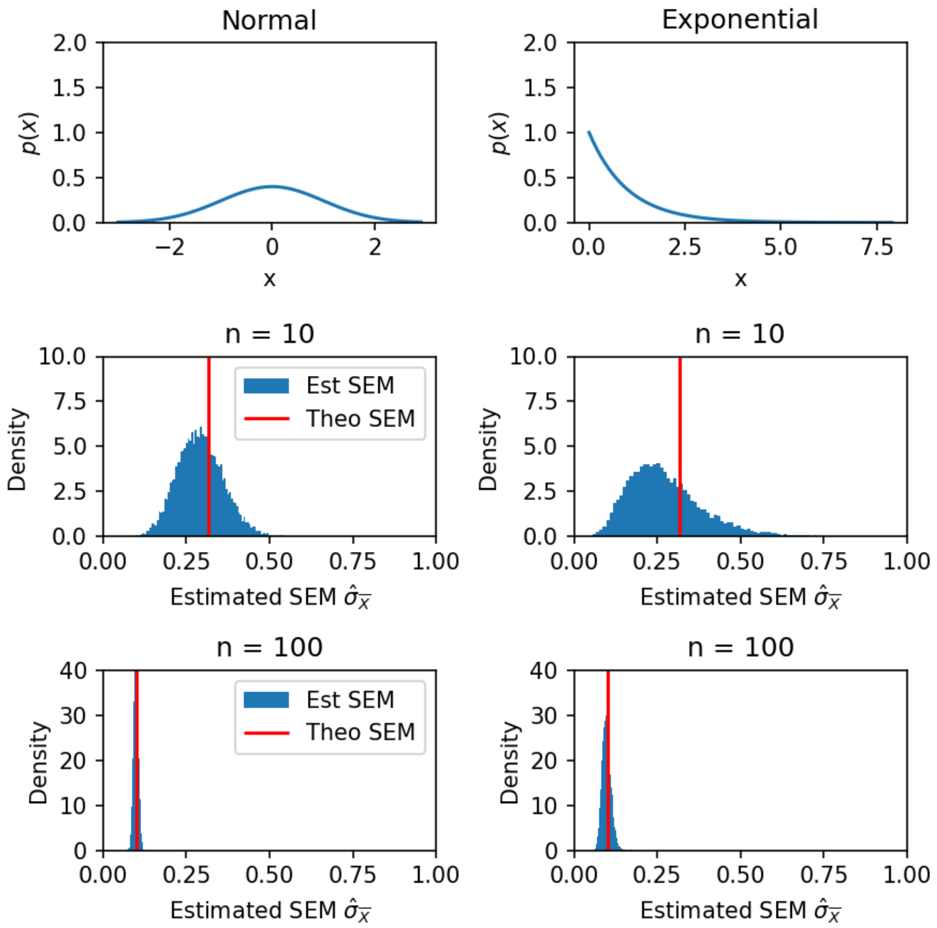
of estimator $\hat{\sigma}_{\hat{\theta}}$

Estimated SEM

$$\hat{\sigma}_{\hat{\mu}} = \frac{S}{\sqrt{n}} \leftarrow \text{r.v.}$$

n large $\Rightarrow$

$$\hat{\sigma}_{\hat{\mu}} \simeq \sigma_{\hat{\mu}}$$



Mean of a large sample

For large samples $\hat{\sigma}_{\hat{\mu}} = S \simeq \frac{\sigma}{\sqrt{n}}$ and so we treat it as a constant.

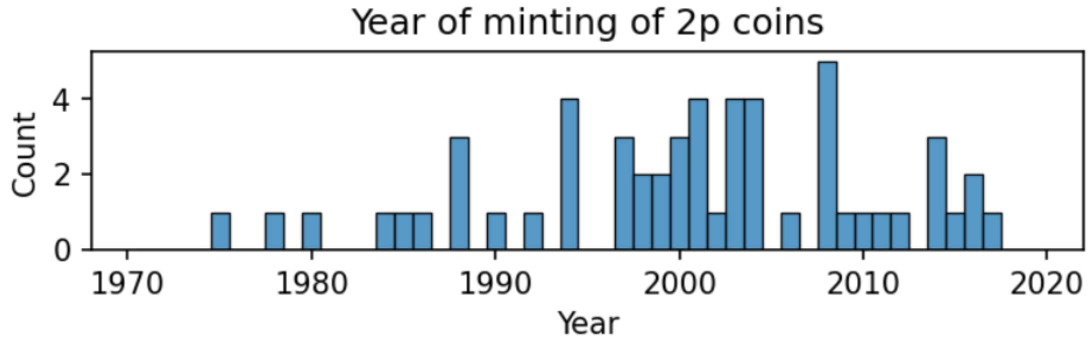
We call the standardised r.v.

$$Z = \frac{\bar{X} - \mu}{S/\sqrt{n}}$$

but since we regard S as fixed, it is a function of $\bar{X}$, and by the CLT is distributed with a standard z - distribution

Confidence interval for the year of a 2p coin

$n = 56$



In a sample of 56 2p coins, the mean year of minting of the 2p coins is 2000.8 and the sample standard deviation is 10.4. Give a 95% confidence interval for the mean year of minting in the population of all 2p coins.

Practice more in next week's workshop sheet

Solution

$n = 56$

Mean age $\bar{x} = 2000.8$ years

Sample st. dev $s = 10.4$ years

S.E.M $\hat{\sigma}_{\bar{x}} = \frac{s}{\sqrt{n}} = \frac{10.4}{\sqrt{56}} = 1.390$

Large sample $(n > 40) \Rightarrow Z = \frac{\bar{X} - \mu}{\hat{\sigma}_{\bar{x}}}$ is z-distributed

95% C.I. $\Rightarrow \alpha = 0.05$

$a = b = z_{\alpha/2} = z_{0.025} = 1.96$

$(\bar{x} - z_{\alpha/2}\hat{\sigma}_{\bar{x}}, \; \bar{x} + z_{\alpha/2}\hat{\sigma}_{\bar{x}})$

$= (2000.8 - 1.96 \times 1.390, \; 2000.8 + 1.96 \times 1.390)$

$= \underline{(1998.1, 2003.5) \text{ is } 95\% \text{ CI}}$

Reporting confidence intervals

$(1998 \;,\; 2004)$

$M = 2001$, $CI = 1998 - 2004$ (95% CI)

$\hat{\mu} = 2001 \pm 3$ (95 % CI)

$\hat{\mu} = 2006.1 \pm 1.4$ (Mean $\pm$ 1. SEM)

Inf2 - Foundations of Data Science:
Estimation –
Interpretation of confidence intervals

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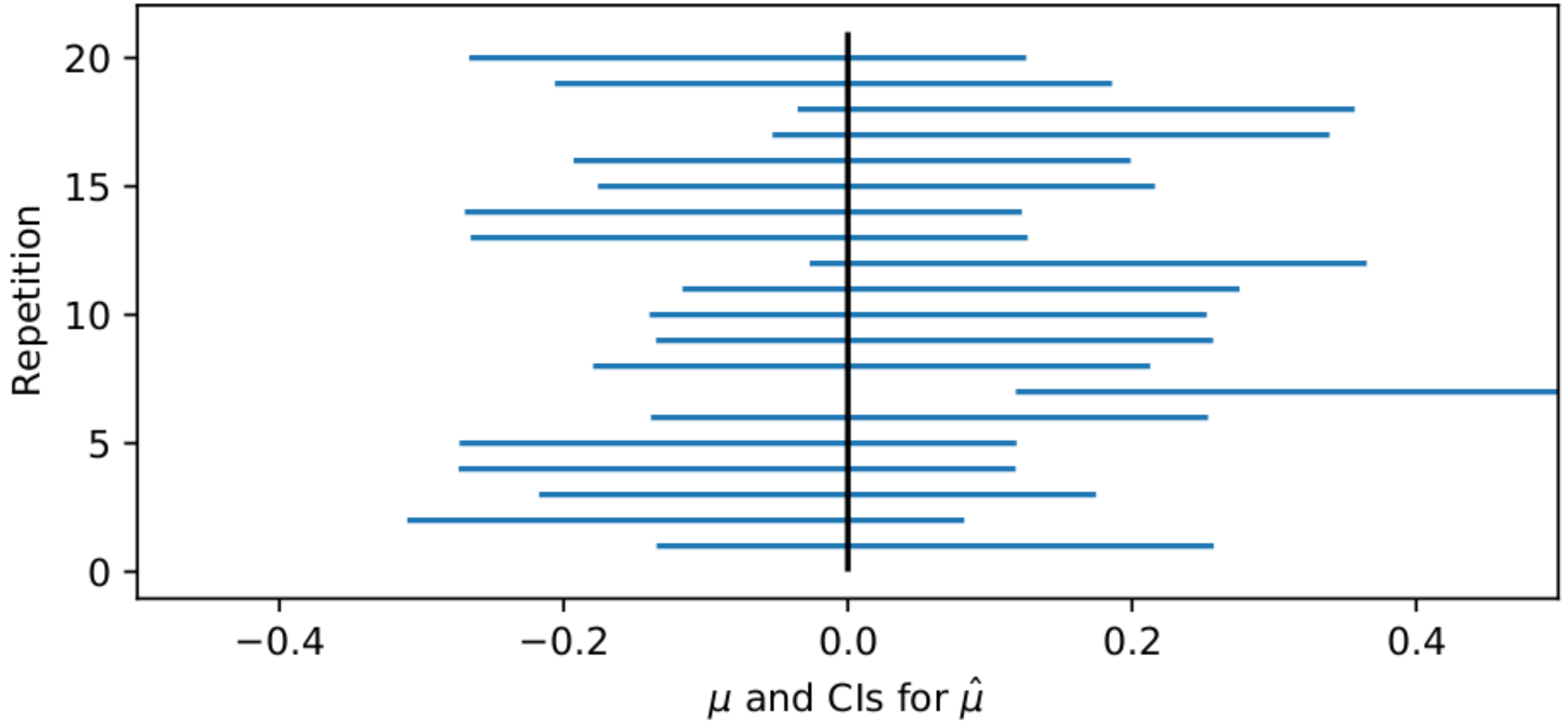
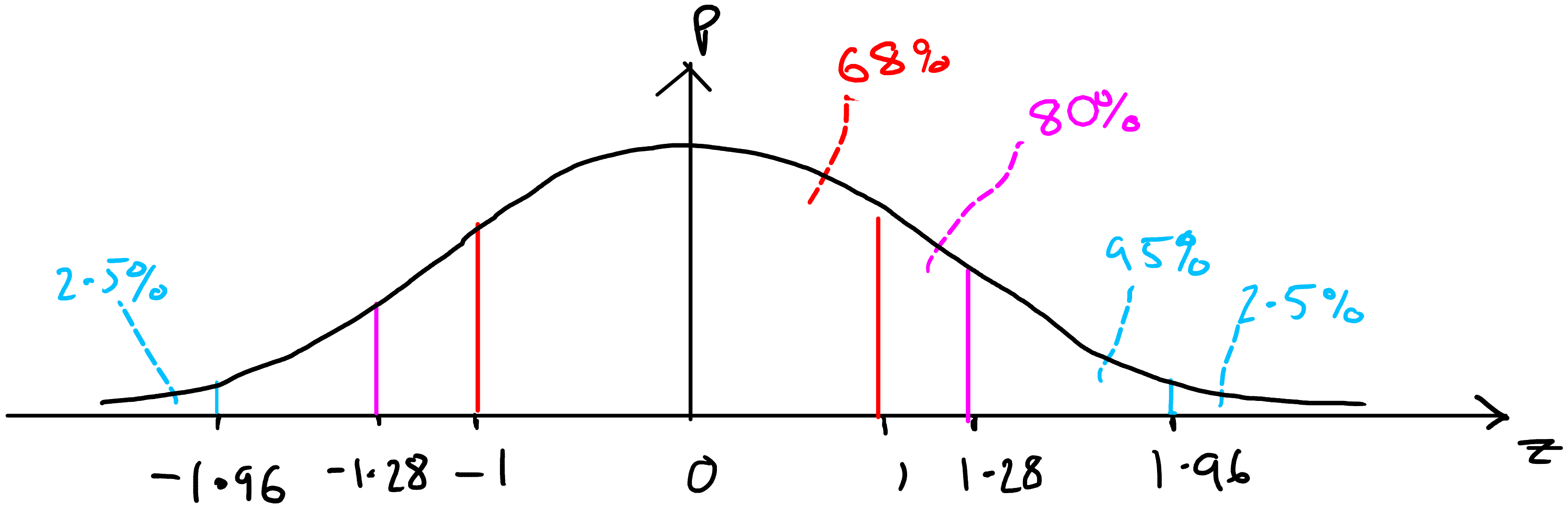
Confidence intervals are a random interval

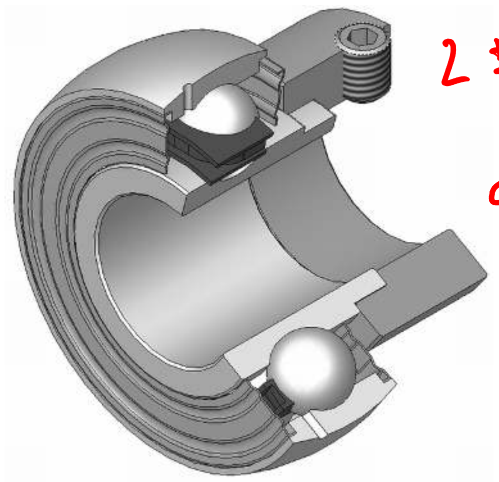
Frequentist interpetation: If we use the same procedure to create a $100(1-\alpha)\%$ confidence interval repeatedly, in the long run the confidence interval will include the true value on $100(1-\alpha)\%$ of repetitions

A given interval either does or does not include the true value, but we don't know, and we shouldn't say it has a $100(1-\alpha)\%$ chance of including the true value



What level of confidence should we choose?





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2 ± 0.00001 mm

99.999 %

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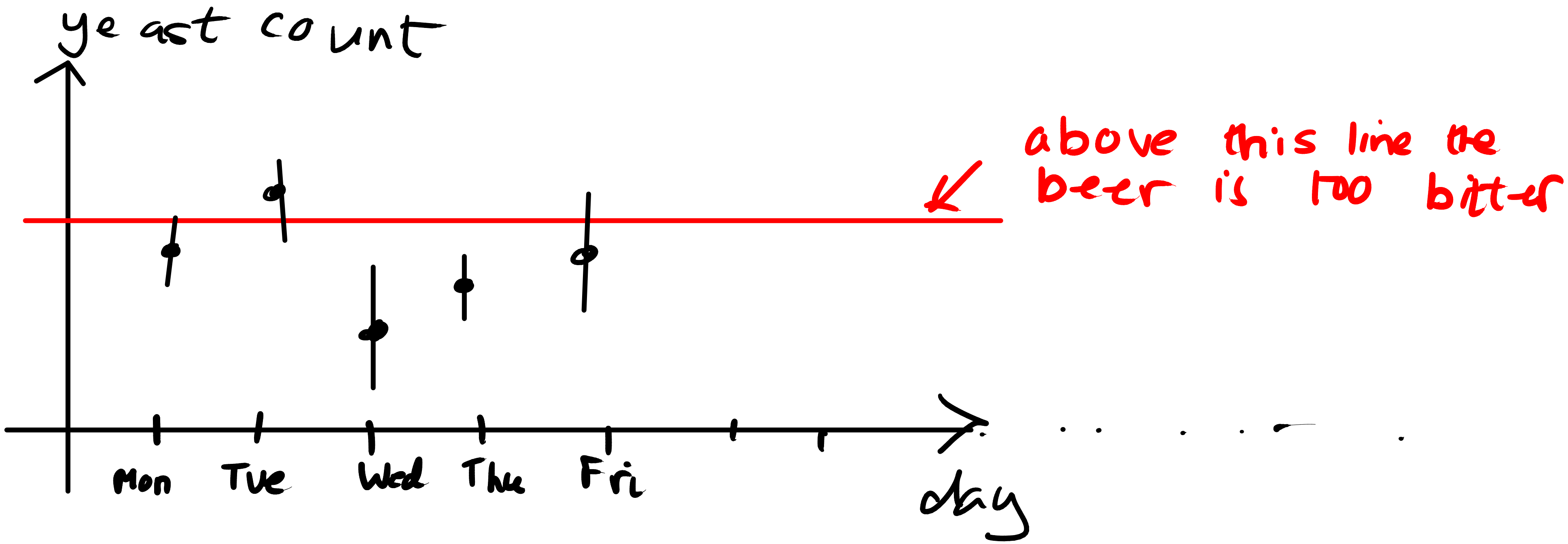
How much data do we collect?

A question inspired by the work of "STUDENT" (aka W. S. Gosset) in a brewery



Suppose we only want 1% of beer to be too bitter
What level of confidence should we have?
How many samples per day should we make?

Inf2 - Foundations of Data Science: Estimation - Bootstrapping

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Principle of bootstrapping

Baron Münchhausen - Wikipedia

- Treat the sample like a population
- Resample estimator from it to get sampling distribution
- Sample is similar to population for a large sample

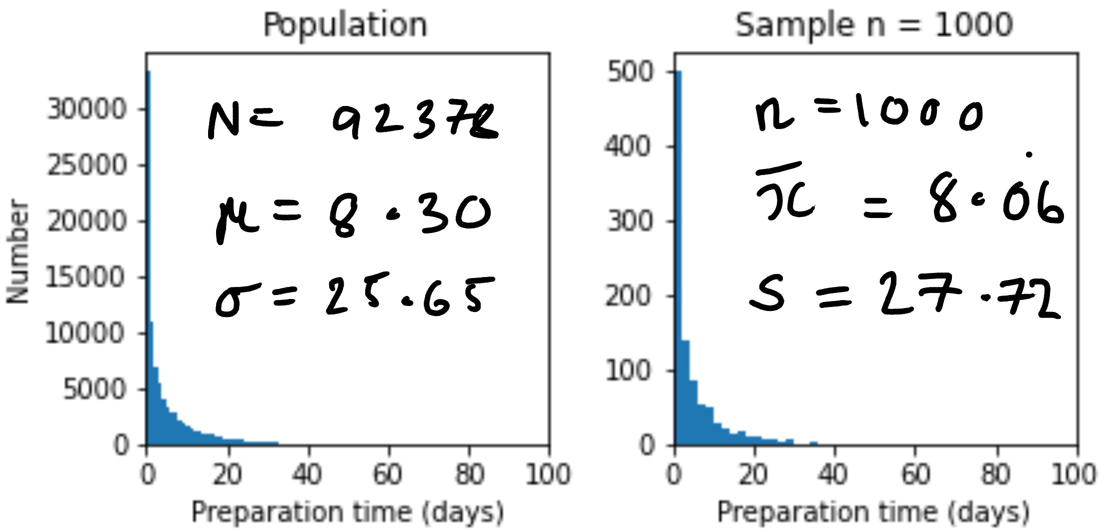
Related lab on the bootstrap

E.g. Japanese restaurant reservation times

Mstyslav Chernov, Wikimedia Commons, CC BY SA 3.0

"Preparation time"
= Time of reservation
  − Time reservation made

2016



	Population	Sample
count	92378.00	1000.00
mean	8.30	8.06
std	25.65	27.72
min	0.00	0.00
25%	0.21	0.17
50%	2.08	1.96
75%	7.88	6.92
max	393.12	364.96

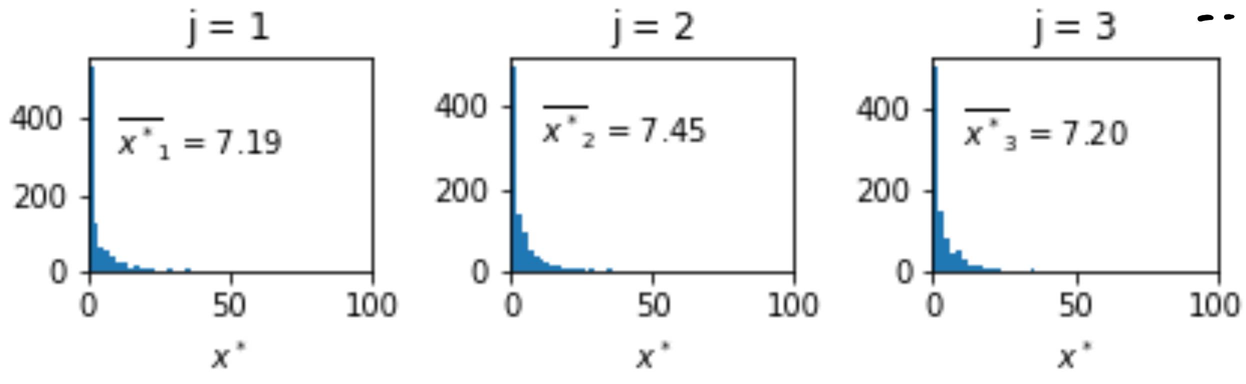
Bootstrap confidence interval for the mean

$n = 1000$ $\quad x$

for $j = 1, \ldots, B$ — # Bootstrap samples

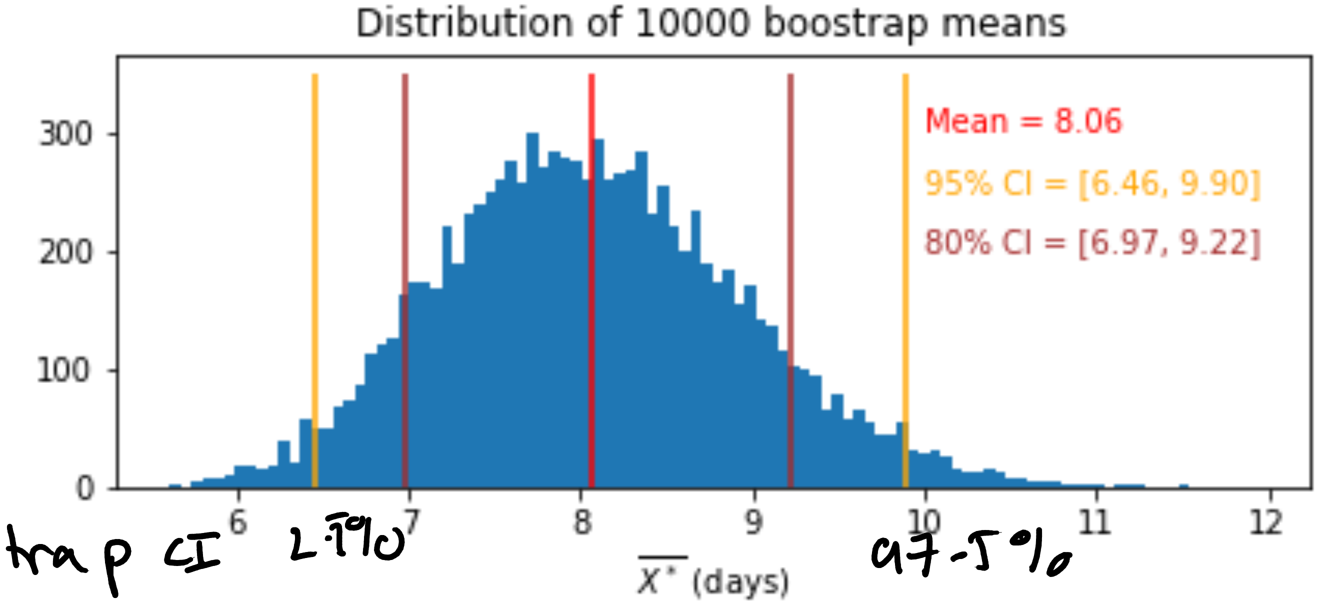
$\quad x^*$ of size n from x with replacement

$\quad \bar{x}^*_j \leftarrow$ mean x^*



... B

$$S^2_{boot} = \frac{\sum_{j=1}^{B} (\bar{x}^*_j - \bar{x})^2}{B-1}$$



(6.46, 9.90) – Bootstrap CI $\quad$ 2.5% $\quad$ 97.5%

(6.34, 9.78) – Normal approx

General formulation of the bootstrap

Bootstrap CI. $\hat{\vartheta}$ — $\hat{\mu}$, $\hat{\sigma}^2$, $\hat{\beta}$

- For j in $1, \ldots, B$
  - Sample n items from x with replacement
  - Compute sample stat of the new sample $\hat{\vartheta}^*_j$
- Bootstrap estimator of variance of statistic

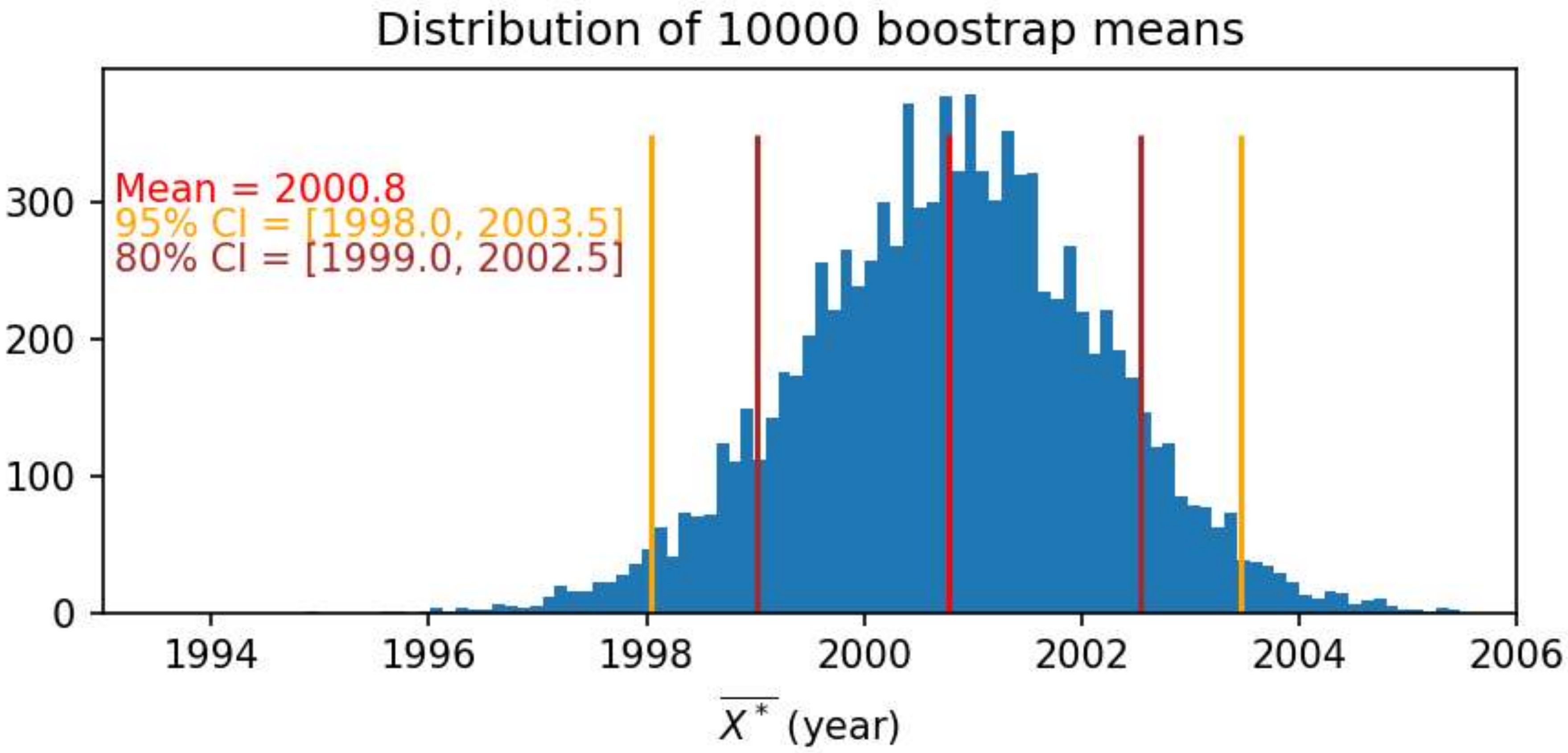
$$s^2_{boot} = \sum_{j=1}^{B} \frac{(\hat{\vartheta}^*_j - \hat{\vartheta})^2}{B-1}$$

- Find CI from Bootstrap dist.

✓ centrality – median mean

✗ Extremes – max or min

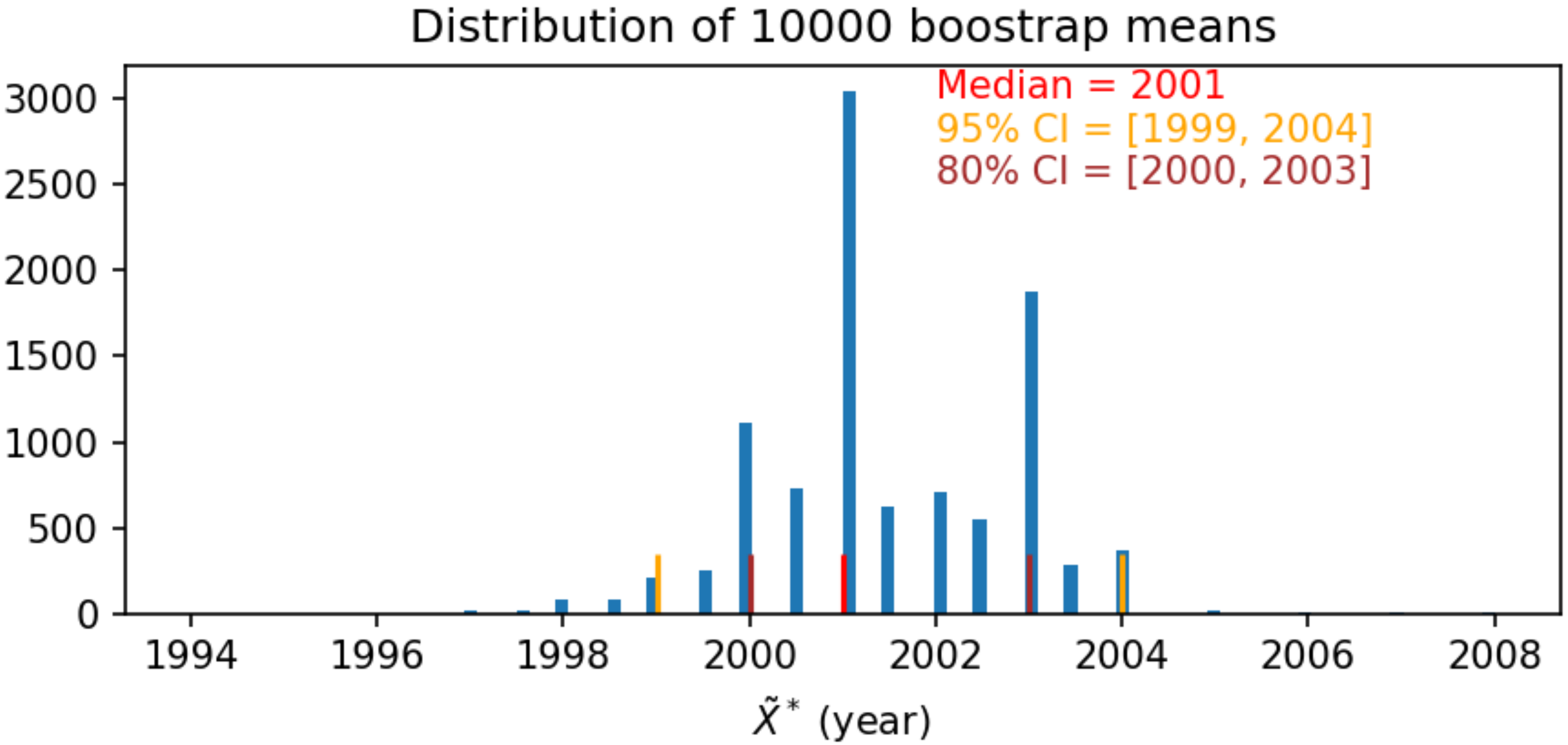
Bootstrap mean coin year



C.f theoretical estimate

(1998.1, 2003.5)

Bootstrap median coin year



Summary

1. Principle and meaning of confidence intervals

2. Confidence intervals of the mean of a large samples ($n > 40$)
   computed theoretically
   - z distribution

3. Confidence intervals of the mean of a small sample ($n < 40$) from a fairly normal distribution computed theoretically
   - t distribtion

4. Confidence intervals for more types of estimator
   computed using the bootstrap