Inf2 - Foundations of Data Science: Hypothesis testing

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Plan for statistical inference

1. Randomness, sampling and simulations (S2 Week 1)

2. Estimation, including confidence intervals (S2 Week 2)

3. Hypothesis testing (S2 Week 3)

4. A/B testing (S2 Week 3)

Onwards to Logistic regression (S2 Week 4)

Today

1. Principle of hypothesis testing (using statistical simulations)

2. p-values (using statistical simulations)

3. Issues in hypothesis testing

4. Theoretical methods

5. Practical applications

6. Example: testing for goodness of fit to a model

Inf2 - Foundations of Data Science:
Hypothesis testing –
Principle of hypothesis testing

Inferential statistics tasks: Hypothesis testing

Yes/no questions:

E.g. 1: "Is Chocolate good for you"

E.g. 2: Is a coin biased? 

E.g. 3: Swain versus Alabama (1965). Is this jury selection procedure biased?

Population of Alabama
26% Black
74% Non-black

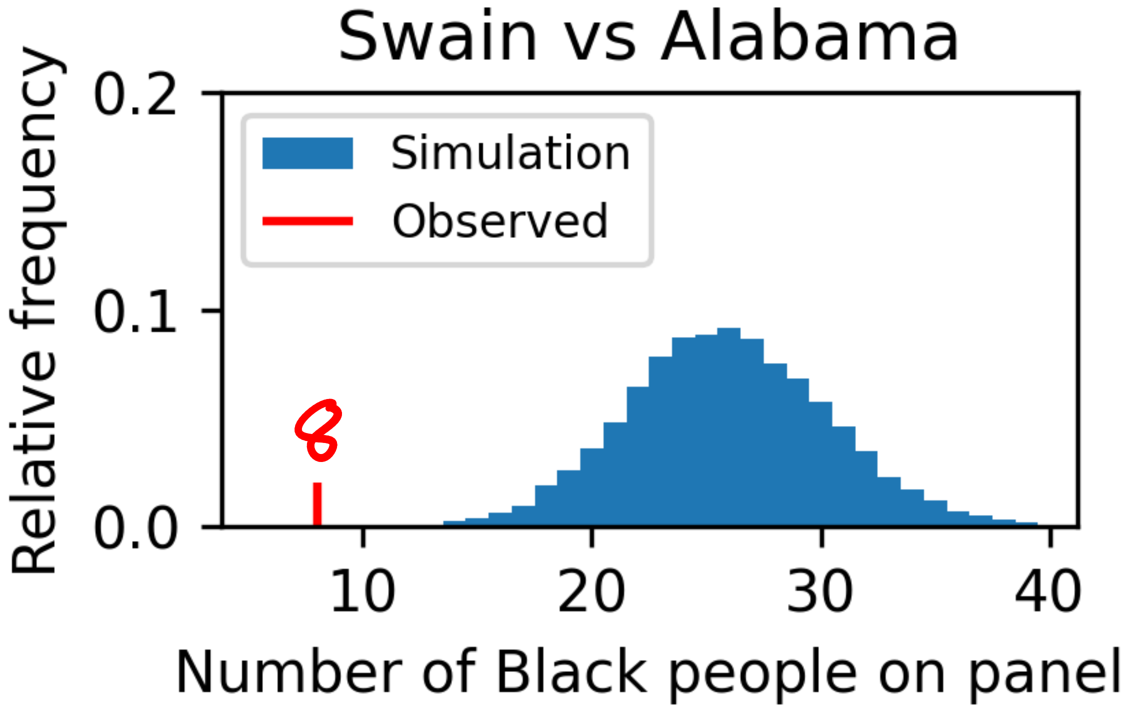
selection procedure

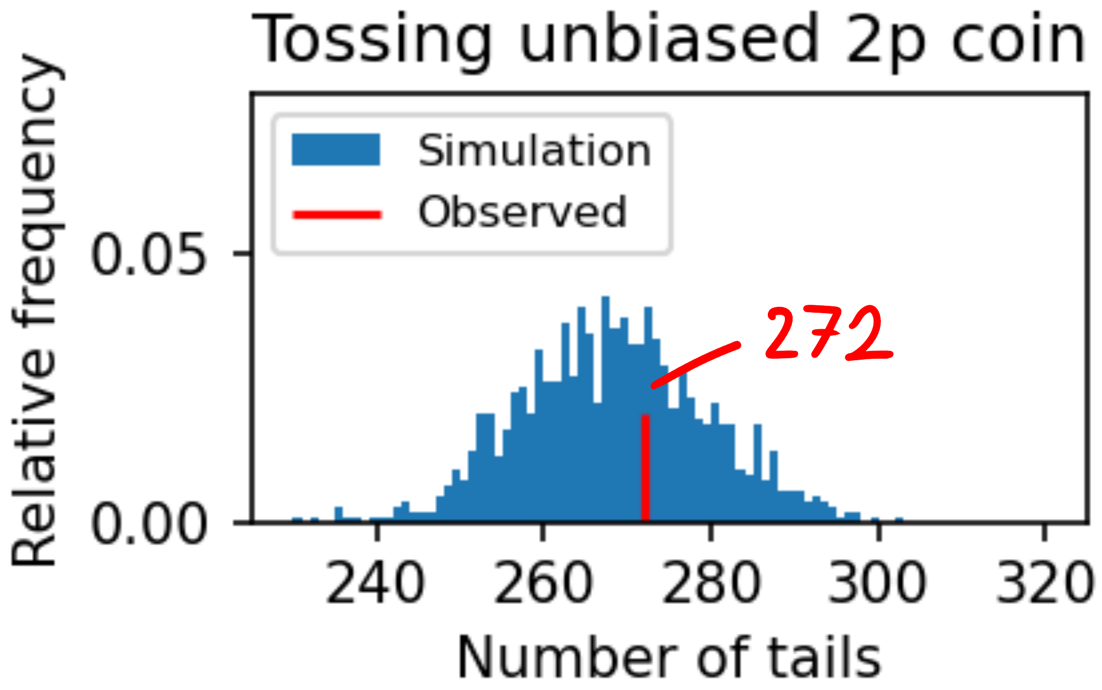
Jury panel of 100:
8 Black and
92 Non-black

Statistical simulation versus observations

Simulate unbiased procedures

Compare with observations





Method of hypothesis testing

Null hypothesis H_0: Claim initially assumed to be true, formalised as a statistical model

e.g. H_0: The jury panel was chosen by random selection from the population in the district.

e.g. H_0 : The coin was unbiased

Alternative hypothesis H_a: Claim contractictory to H_0, typically not formalised as a statistical model

e.g. H_a: The jury was chosen by some other, unspecified, method [that was unfavourable to Black people]

e.g. H_a : The coin is biased (either towards heads or tails)

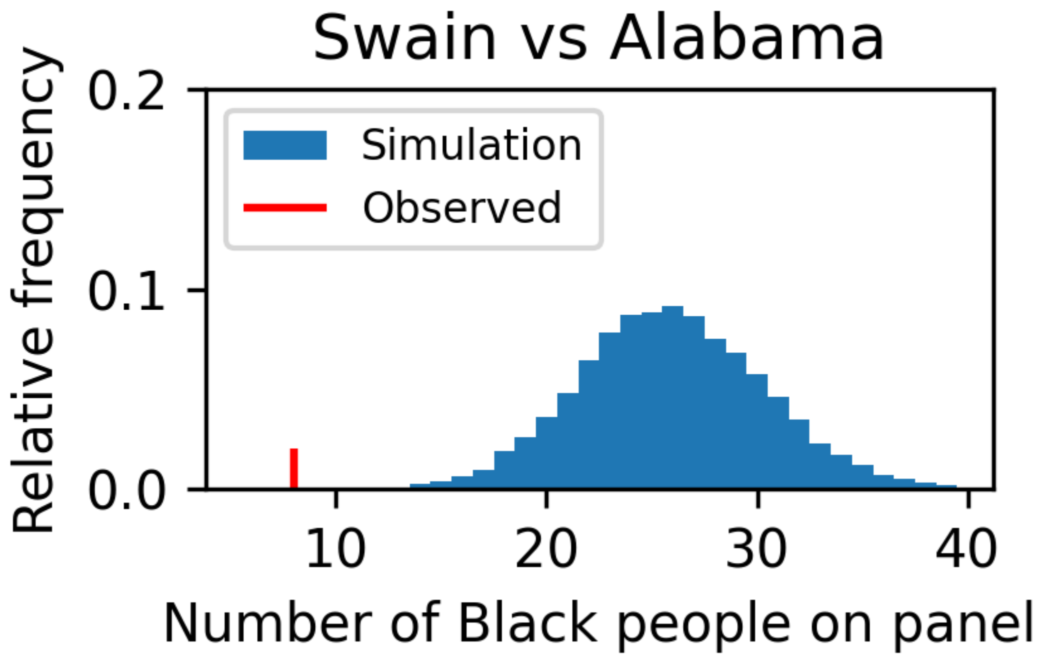
AIM: Reject or not reject H_0

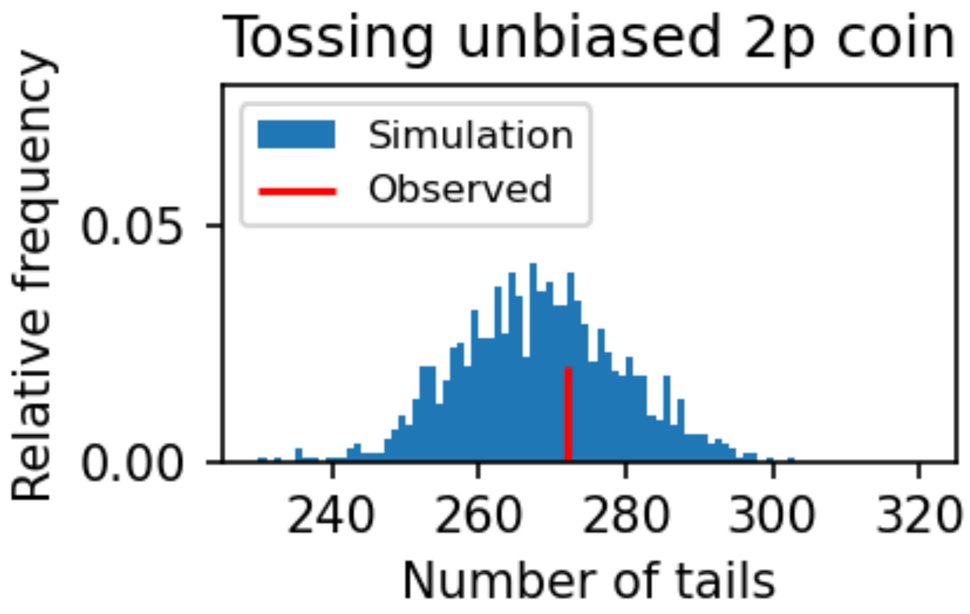
Test procedure

1. Test statistic: e.g. number of black people on a jury panel

   $t_0 = 8$ (observed)

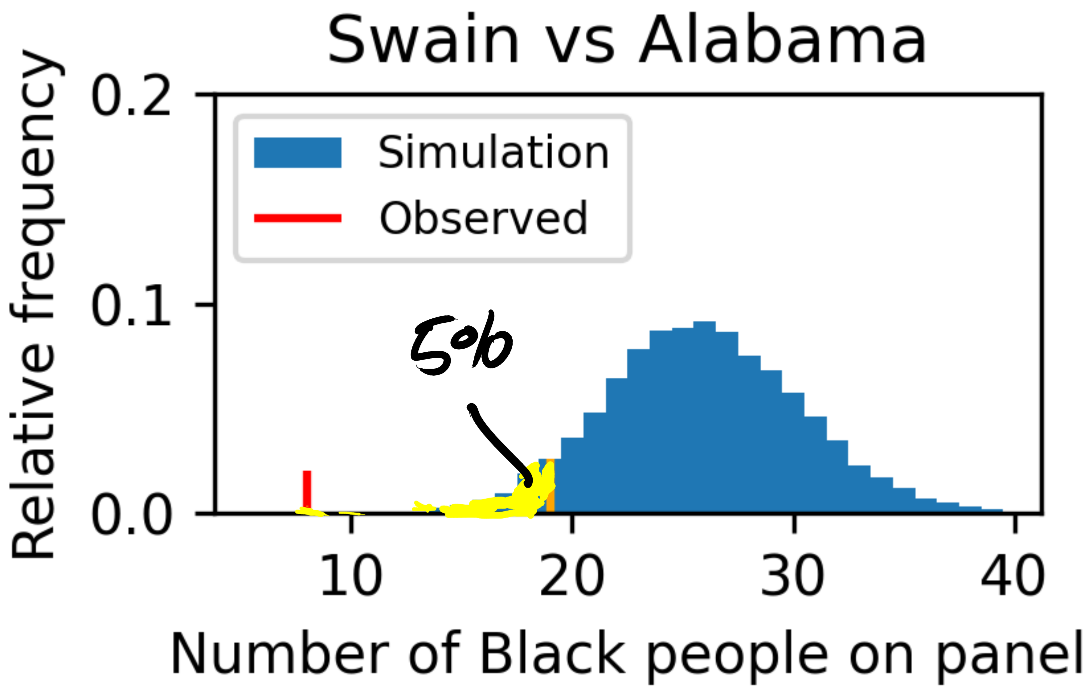
2. Distribution of the test statistic under H_0





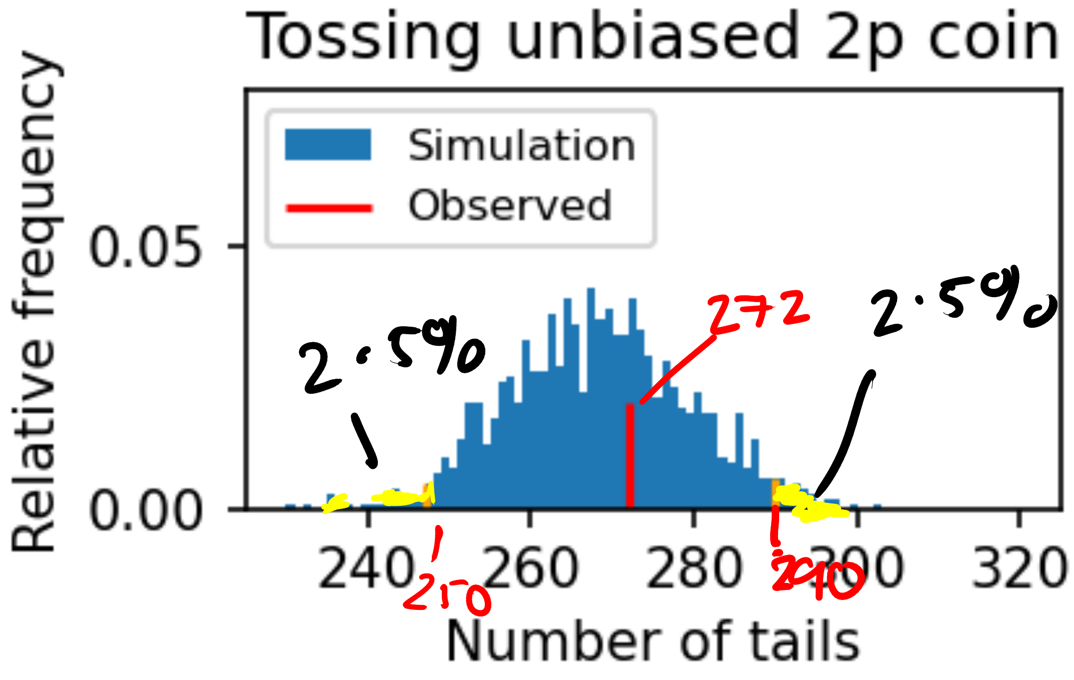
3. (a) Rejection region
   (b) Return a p-value

One-tailed rejection regions



H_a: Number of black people is below the number expected by chance

Two-tailed rejection regions



H_a: Number of tails is different from the number expected by chance

Observation in rejection region => reject, otherwise do not reject

Reject at 5% level?

Reject at 5% level?

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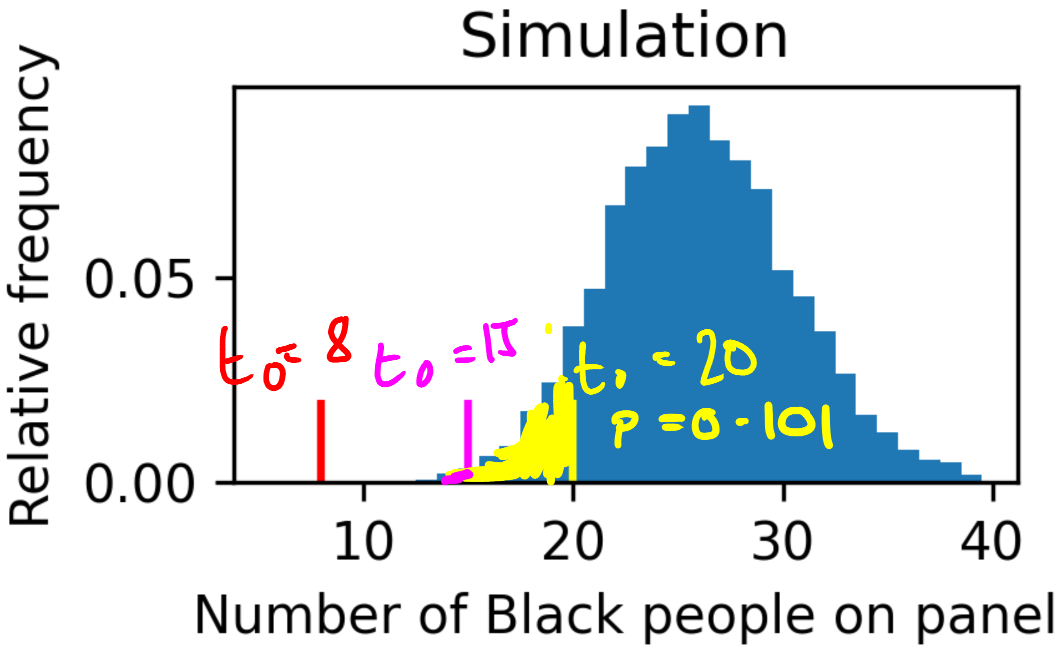
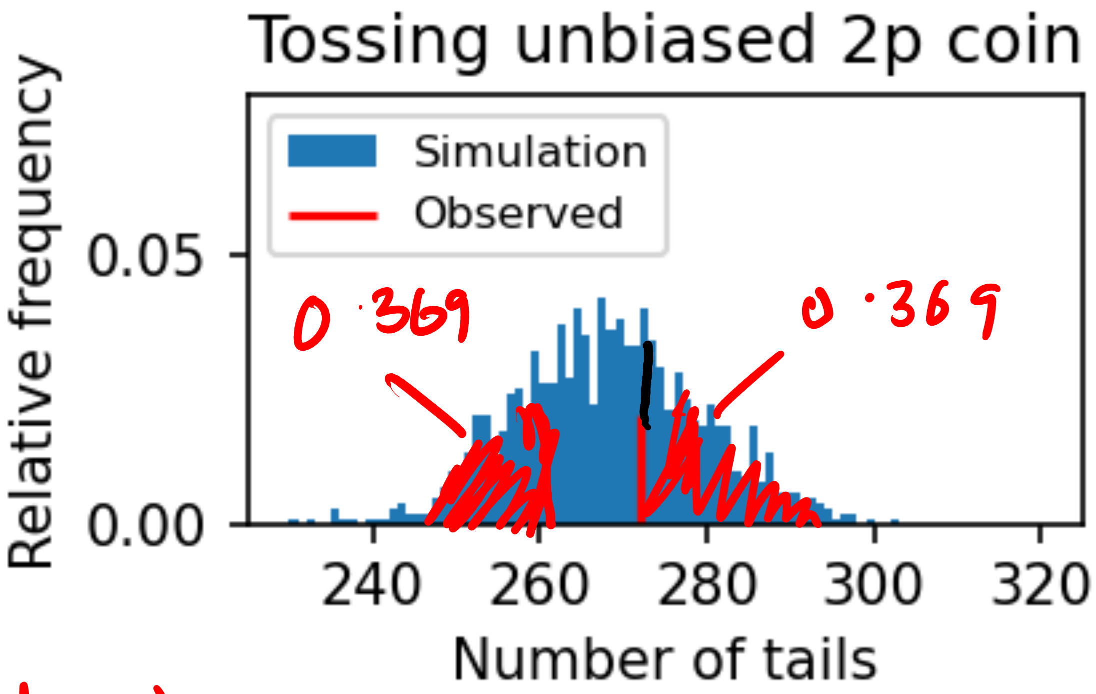
Principle of p-values

Observed data is boundary of rejection region





$p = 0.101 > 0.05$ (one tailed test)

$t_0 = 15 \Rightarrow p = 0.0062 < 0.01$ (one tailed test)

$p = 2 \times 0.369 = 0.738$

0.369

Definitions of the p-value

THE AMERICAN STATISTICIAN
2016, VOL. 70, NO. 2, 129–133
http://dx.doi.org/10.1080/00031305.2016.1154108

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EDITORIAL

The ASA's Statement on *p*-Values: Context, Process, and Purpose

Informally, a p-value is the probability under a specified statistical model that a statistical summary of the data (e.g., the sample mean difference between two compared groups) would be equal to or more extreme than its observed value.

The p-value is the probability, calculated assuming the null hypothesis is true, of obtaining a value of the test statistic at least as contradictory to H0 as the value calculated from the available sample.
(Modern Mathematical Statistics with Applications, p. 456)

Question

1. In the hypothetical case of 8 black people on the jury, which has a p-value of 0.10, is the null hypothesis true?

2. For the coin tossing, is the proability that 2p coins are unbiased equal to

p=0.738

or

1-p = 0.262 ?

What p-values are and are not (ASA Statement on Statistical Significance and P-values)

P-values can indicate how incompatible the data are with a specified statistical model.

low p $\Rightarrow$ high incompatibility $\Rightarrow$ Evidence against H_0

P-values do not measure the probability that the studied hypothesis is true, or the probability that the data were produced by random chance alone.

Role of hypothesis testing

1. Decide whether a hypothesis or model is compatible with data from observational studies or randomised experiments

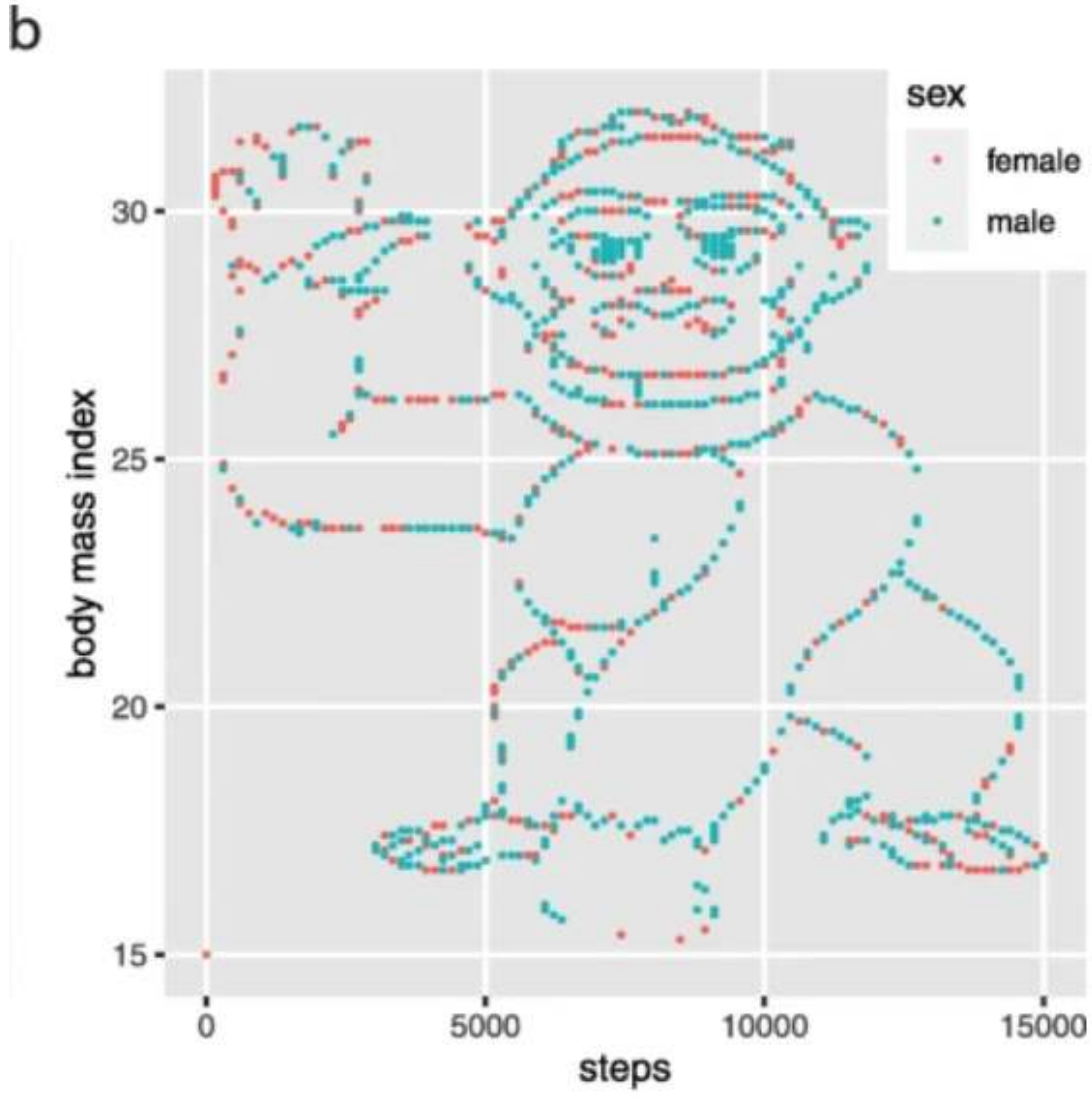
2. Investigate mechanisms specific to data

Question

Suppose you are asked to investigate if there is a relationship between BMI (body mass index) and steps walked.

This data visualised apparently shows the relationship between BMI and steps walked each day by men and women.

Having seen this visualisation would you this data to test if there is a relationship between BMI and number of steps walked?



Yanai and Lercher (2020)

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"Statistical significance"

$p < 0.05 \Rightarrow$ "statistically significant"

*	significant at the $p < 0.05$ level
**	" " " $p < 0.01$ "
***	" " " $p < 0.001$ "

Q: Why do so many colleges and grad schools teach p=0.05?
A: Becuase that's still what the scientific community and journal editors use

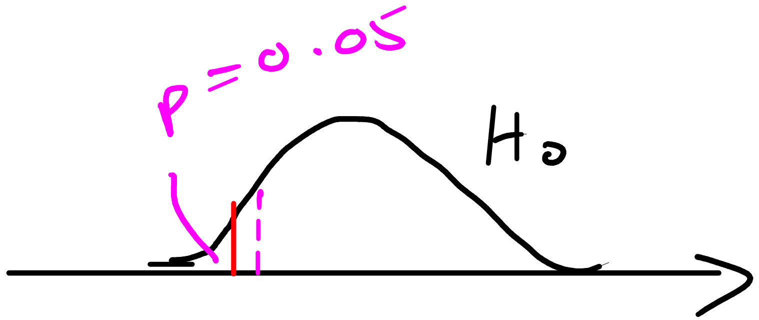
Q: Why to so many people still use p=0.05?
A: Becuase that's what they were taught at grad school.

- George Cobb, ASA Statement on p-values

Question

In the coin tossing experiment, imagine that we repeat the experiment 1000 times and that we demand statistical significance at the 0.01 level. Assuming the null hypothesis is true (unbiased coin), on how many experiments do we expect to reject the null hypothesis?

Type I and Type II Errors



Type I error: Rejecting H_0 when it is true

- control by setting α-size of rejection region

Type II error: not rejecting H_0 when it is false

- more difficult to control for

Proper inference requires full reporting and transparency

- P-values and related analyses should not be reported selectively.

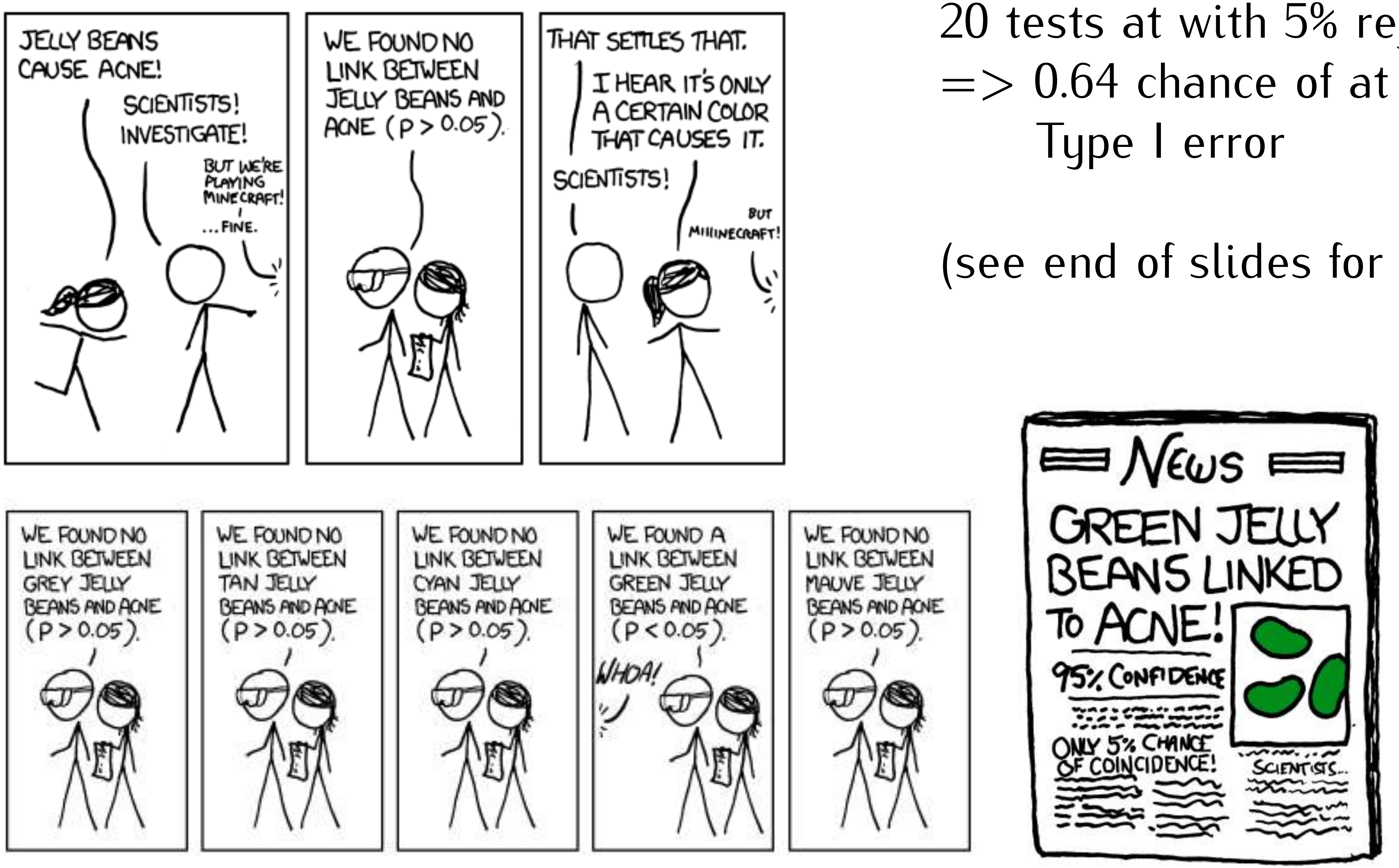
- Conducting multiple analyses of the data and reporting only those with certain p-values (typically those passing a significance threshold) renders the reported p-values essentially uninterpretable.

- Cherry-picking promising findings, also known by such terms as
  - data dredging,
  - significance chasing, significance questing, selective inference,
  - and "p-hacking,"

  leads to a spurious excess of statistically significant results in the published literature and should be vigorously avoided. . .

(ASA Statement on Statistical Significance and P-values)

Multiple testing



20 tests at with 5% rejection region
=> 0.64 chance of at least one
Type I error

(see end of slides for explanation)

https://www.explainxkcd.com/wiki/index.php/882:_Significant

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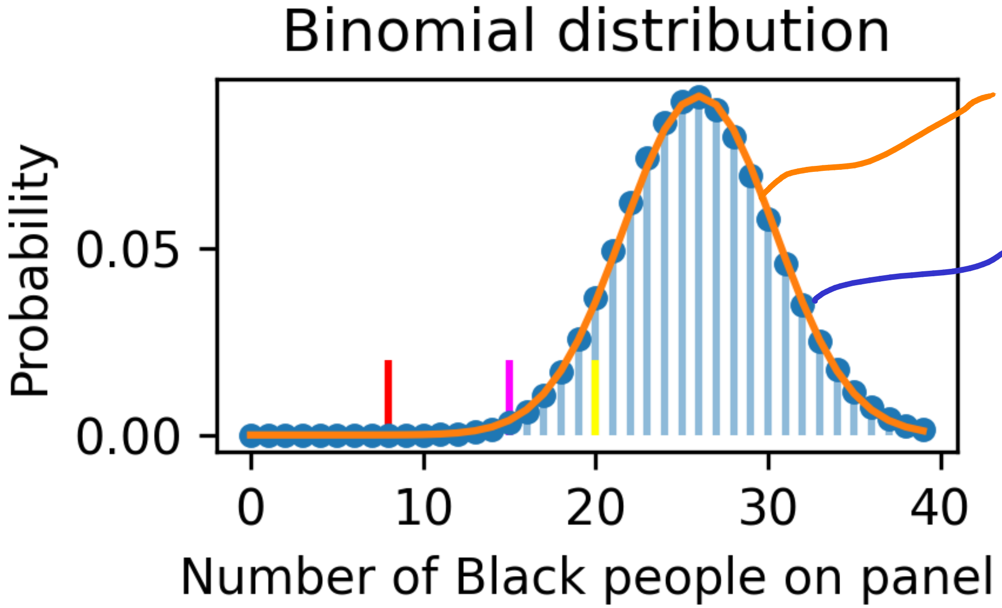
Determining p-values from probability dists



Normal dist

$\mu = np$

$\sigma = \sqrt{np(1-p)}$

$b(t_0; n, p)$

$n = 100$

$p = 0.26$

Binomial

cumulative dist.

$$p\text{-value} = P(T_0 \le t_0) = B(t_0; n, p) = \sum_{t=0}^{t_0} b(t; n, p)$$

Normal approximation

$$p\text{-value} = \Phi\left(\frac{t_0 - \mu}{\sigma}\right)$$

where $\Phi(z)$ cumulative dist function of z-distribution

Normal approximation to the binomial distribution

n large $\Rightarrow$ binomial dist is approx normal with $\mu = np$ and $\sigma^2 = np(1-p)$

$\Rightarrow$ $Z = \dfrac{T_0 - \mu}{\sigma}$ has a z-distribution

1% rejection region has 99% of weight to its right $\Rightarrow$

At boundary of 1% rejection region

$$Z = z_{0.99} = \frac{T_0 - \mu}{\sigma} \Rightarrow T_0 = \mu + z_{0.99}\,\sigma$$

$\mu = np = 100 \times 0.26 = 26$ $\quad \sigma = \sqrt{np(1-p)} = \sqrt{100 \times 0.26 \times (1 - 0.26)}$

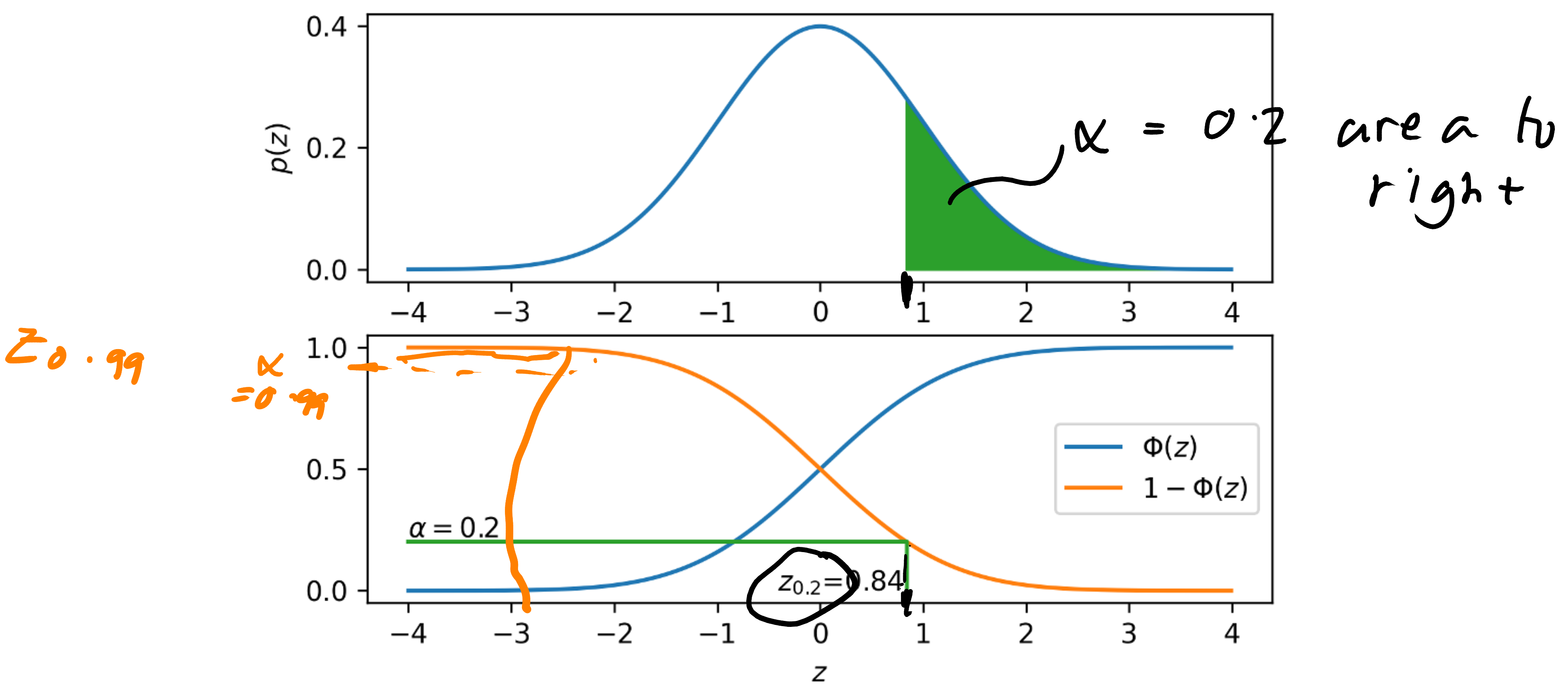
$= 4.386$

$z_{0.99} = -z_{0.01} = -2.326$

$\Rightarrow$ Boundary of 1% (lower tailed) rejection region $T_0 = 26 - 2.326 \times 4.386$

$T_0 = 15.8$

z-critical values



P-values computed by various methods for Swain versus Alabama

t_0	Simulation	Binomial	Normal
8	0	4.73e-06	2.03e-05
15	0.0067	0.0061	0.0061
20	0.1020	0.1030	0.0857

Inf2 - Foundations of Data Science: Hypothesis testing - Testing for goodness-of-fit

Multiple categories

American Civil Liberties Union investigation into jury selection in Alameda County, CA

	Caucasian	Black/AA	Hispanic	Asian/PI	Other	Total
Population %	54	18	12	15	1	100
Observed panel numbers	780	117	114	384	58	1453
Expected panel numbers	784.62	261.54	174.36	217.95	14.53	1453.00
$\frac{(\text{Observed}-\text{Expected})^2}{\text{Expected}}$	0.03	79.88	20.90	126.51	130.05	357.36

χ^2 - test statistic

H_0: The panels were chosen by random selection from the population

H_a: The panels were chosen by some other, unspecified method.

1. Test statistic

k - groups

p_i - population proportion in the ith group

n_i - observed number in ith group

n - total size of population $n = \sum_{i=1}^{k} n_i$

np_i - expected number in each group.

obs exp

$$\chi^2 = \sum_{i=1}^{k} \frac{(n_i - np_i)^2}{np_i}$$

e.g.

$np_i = 100 \quad \overbrace{\quad n_i = 95}^{5}$ 5%

$np_i = 10 \quad \underbrace{\quad n_i = 5}_{5}$ 50%

$\chi^2 = 357.36$

"chi-squared"

Generally used to measure goodness-of-fit.

2. H_0 formulated as a statistical model

Draw $n_1, \dots, n_k$ from Multinomial distribution

$$p(n_1, \dots n_k) = n! \, \frac{p_1^{n_1} \cdot \dots \cdot p_k^{n_k}}{(n_1!) \dots (n_k!)}$$

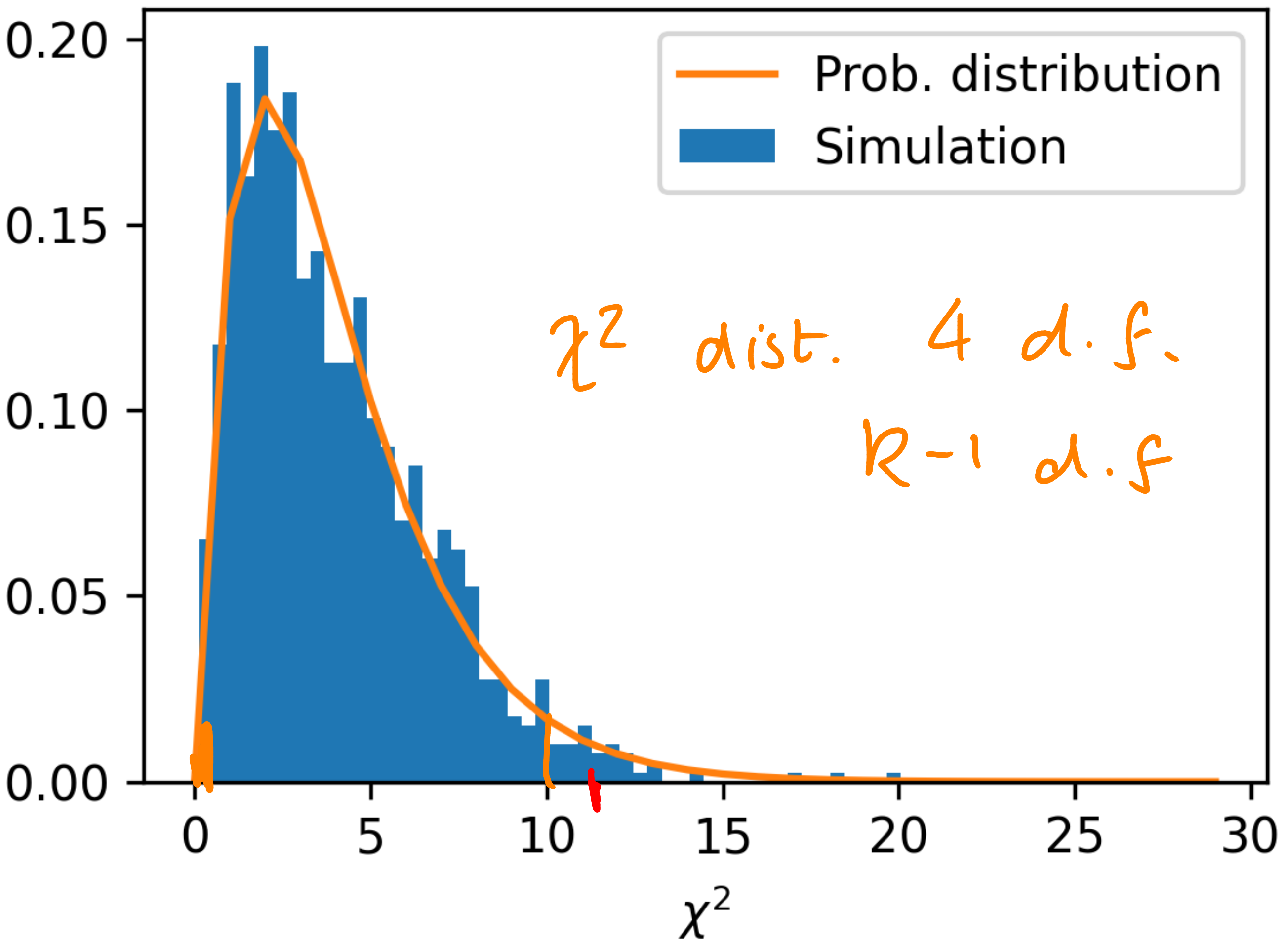
but constrained so that $\sum_{i=1}^{k} n_i = n$

$\Rightarrow$ $k-1$ degrees of freedom.

CODE

2. Distribution of test statistic under H_0

$n = 1543$



$\rightarrow 357$

$p \simeq 0$

Statistical tables

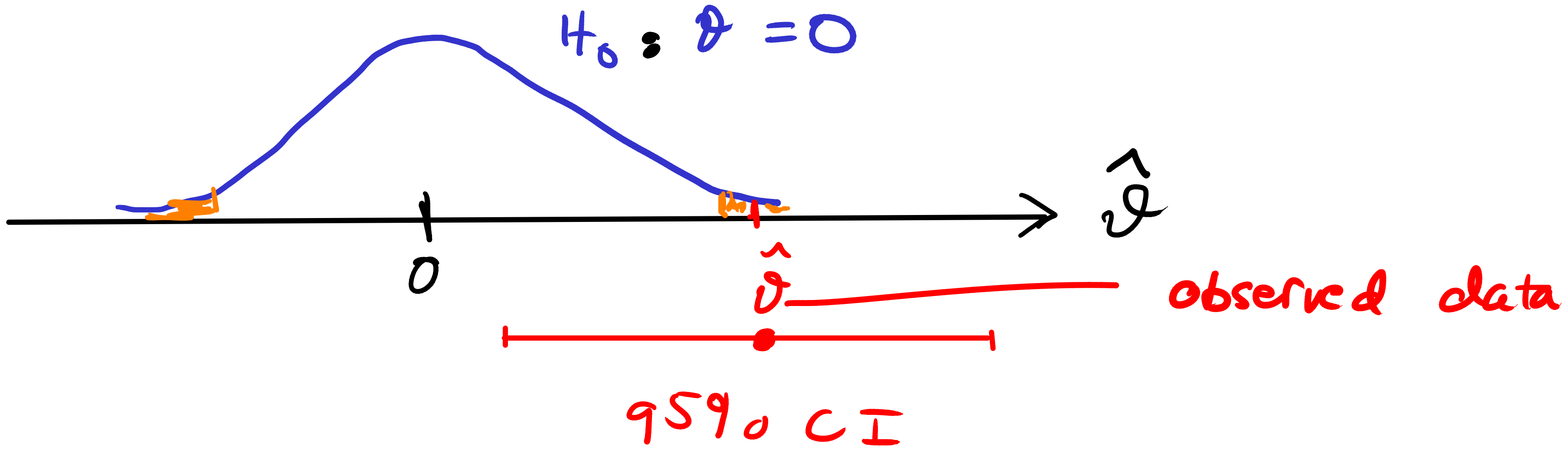
Table A.10 Chi-Squared Curve Tail Areas

d.f

Upper-Tail Area	$\nu = 1$	$\nu = 2$	$\nu = 3$	$\nu = 4$	$\nu = 5$
> .100	< 2.70	< 4.60	< 6.25	< 7.77	< 9.23
.100	2.70	4.60	6.25	7.77	9.23
.095	2.78	4.70	6.36	7.90	9.37
.090	2.87	4.81	6.49	8.04	9.52
.085	2.96	4.93	6.62	8.18	9.67
.080	3.06	5.05	6.75	8.33	9.83
.075	3.17	5.18	6.90	8.49	10.00
.070	3.28	5.31	7.06	8.66	10.19
.065	3.40	5.46	7.22	8.84	10.38
.060	3.53	5.62	7.40	9.04	10.59
.055	3.68	5.80	7.60	9.25	10.82
.050	3.84	5.99	7.81	9.48	11.07
.045	4.01	6.20	8.04	9.74	11.34
.040	4.21	6.43	8.31	10.02	11.64
.035	4.44	6.70	8.60	10.34	11.98
.030	4.70	7.01	8.94	10.71	12.37
.025	5.02	7.37	9.34	11.14	12.83
.020	5.41	7.82	9.83	11.66	13.38
.015	5.91	8.39	10.46	12.33	14.09
.010	6.63	9.21	11.34	13.27	15.08
.005	7.87	10.59	12.83	14.86	16.74
.001	10.82	13.81	16.26	18.46	20.51
p < .001	> 10.82	> 13.81	> 16.26	> 18.46	> 20.51
Upper-Tail Area	$\nu = 6$	$\nu = 7$	$\nu = 8$	$\nu = 9$	$\nu = 10$
> .100	< 10.64	< 12.01	< 13.36	< 14.68	< 15.98
.100	10.64	[illegible]	[illegible]	[illegible]	[illegible]

Inf2 - Foundations of Data Science: Hypothesis testing - Confidence intervals and p-values

Confidence intervals and p-values

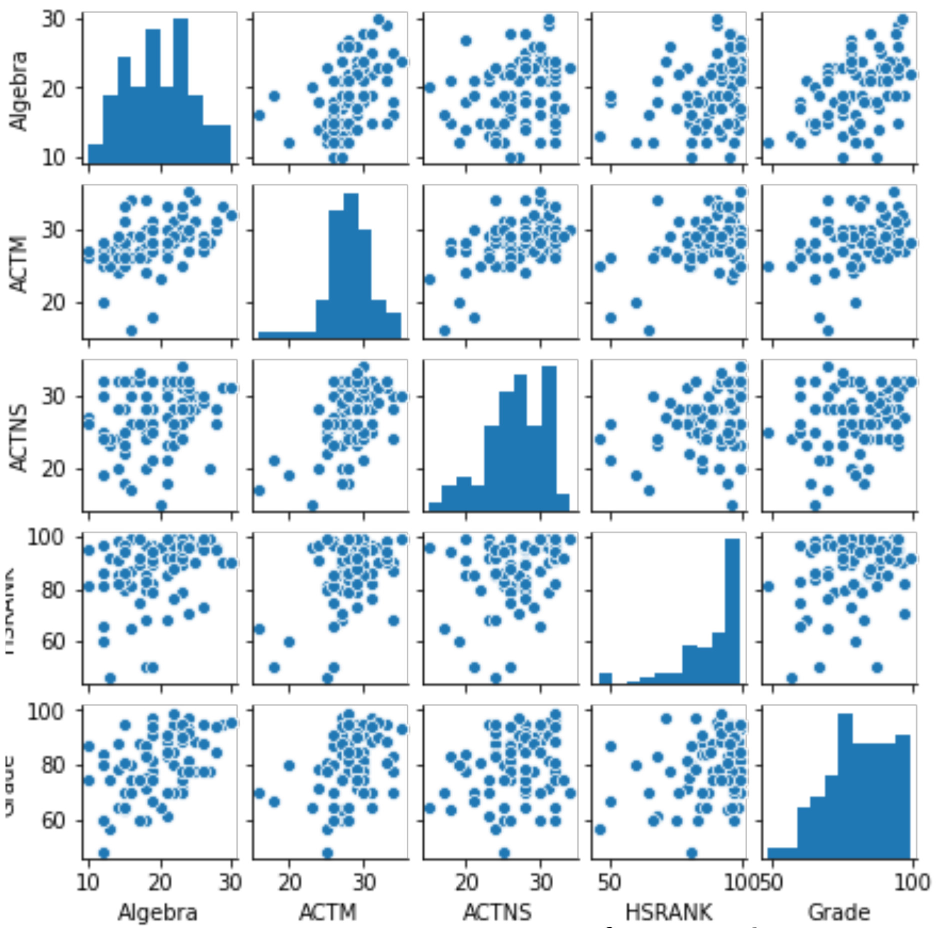


Approx relation: if 95% CI for a parameter doesn't contain 0, reject H_0 that $\vartheta = 0$

p-values in Regression output

Dep. Variable:	Grade	**R-squared:**	0.289
Model:	OLS	**Adj. R-squared:**	0.251
Method:	Least Squares	**F-statistic:**	7.622
Date:	Wed, 26 Oct 2022	**Prob (F-statistic):**	3.30e-05
Time:	09:42:47	**Log-Likelihood:**	-294.31
No. Observations:	80	**AIC:**	598.6
Df Residuals:	75	**BIC:**	610.5
Df Model:	4		
Covariance Type:	nonrobust		

	coef	**std err**	**t**	**P>|t|**	**[0.025**	**0.975]**
Intercept	36.1215	10.752	3.360	0.001	14.703	57.540
Algebra	0.9610	0.264	3.640	0.000	0.435	1.487
ACTM	0.2718	0.454	0.599	0.551	-0.632	1.175
ACTNS	0.2161	0.313	0.690	0.492	-0.408	0.840
HSRANK	0.1353	0.104	1.306	0.196	-0.071	0.342



$x^{(1)}$ $x^{(2)}$ $x^{(3)}$ $x^{(4)}$ y

Edge and Friedberg (1984)

Summary

1. Principle of Hypothesis testing
   (a) Rejection method
   (b) p-values

2. Hypothesis testing applied to problems involving
   testing if observed numbers are consistent with
   expected proportions
   - Many other uses

3. Uses and limitations of hypothesis testing and p-values

Working for multiple-testing example

Suppose 20 tests ; 0·05 chance Type I error on each test

$\Rightarrow$ 0·95 chance of no type I error one test

$\Rightarrow$ $0 \cdot 95^{20}$ chance no type I errors overall

$\Rightarrow$ $1 - 0 \cdot 95^{20} = 0 \cdot 64$ chance type I error