Inf2 - Foundations of Data Science: Regression and inference - Generalised linear models

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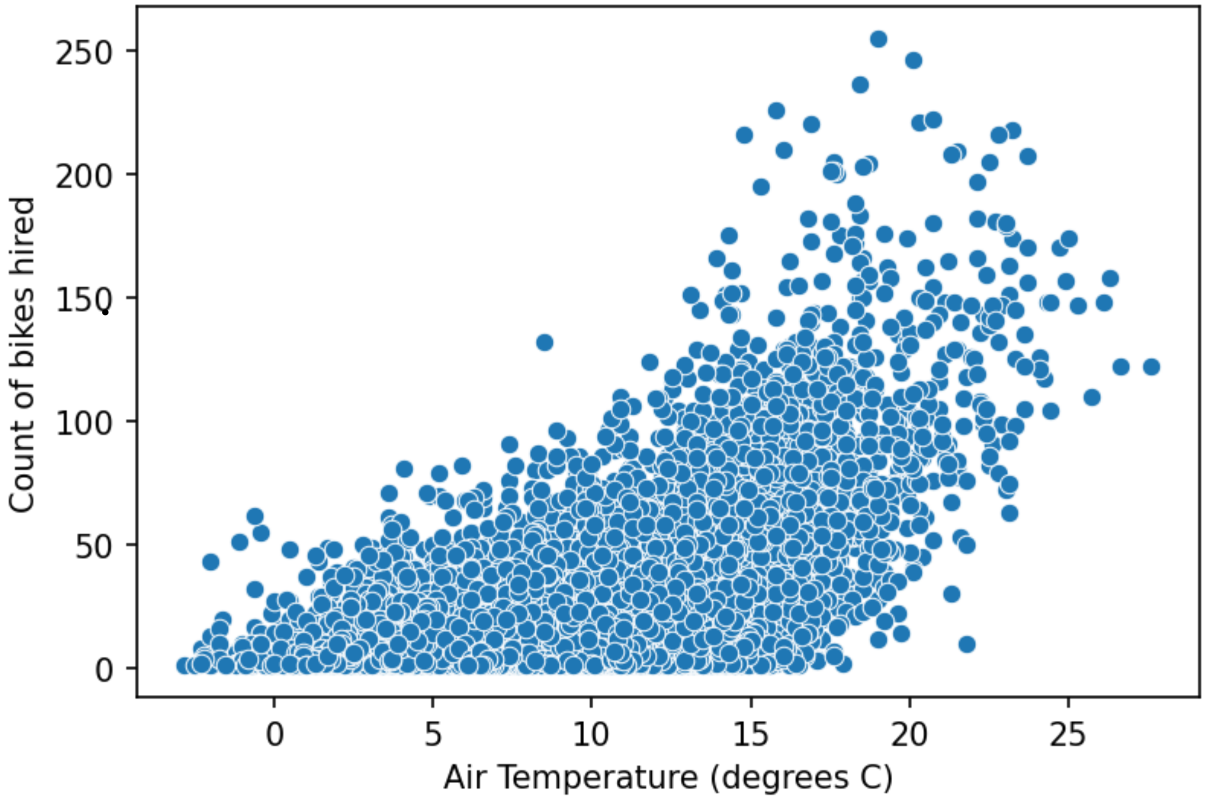
We want to investigate the relationship between the number of bikes hired in an hour and the mean temperature during that hour

Is there a problem with using ordinary least squares linear regression to do this?

Image copyright Pashley Cycles

Data sources:
- Edinburgh Just Eat Bikes data 2020
- Edinburgh temperature observations, Met Office via MIDAS



Overview

Monday

1. The maximum likelihood principle
2. Application of max likelihood to a simple example
3. Application of max likelihood to linear regression

Today

0. Recap
1. Max likelihood with non-normal distributions
2. Poisson regression
3. Logistic regression and generalised linear models

Inf2 - Foundations of Data Science: Regression and inference - Recap of max likelihood applied to linear regression

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Likelihood and log likelihood as a function of parameters

$P(Y = y_1, \ldots, y_n \mid \mu, \sigma^2)$

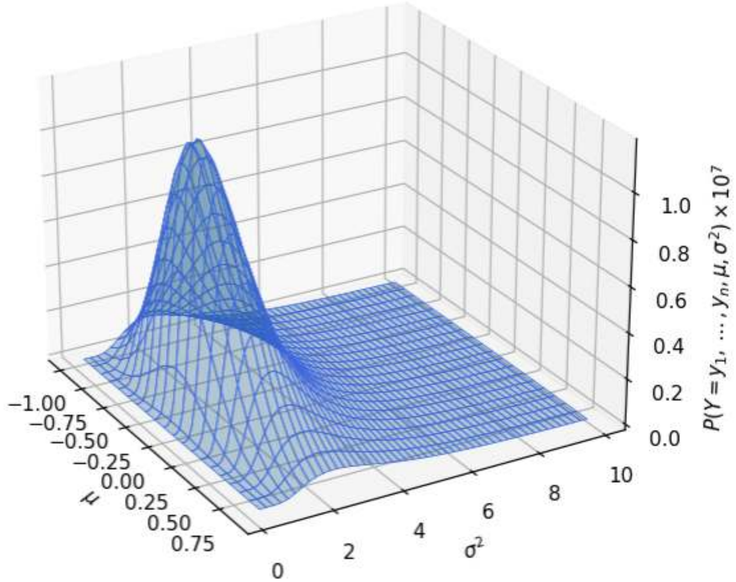
Data:

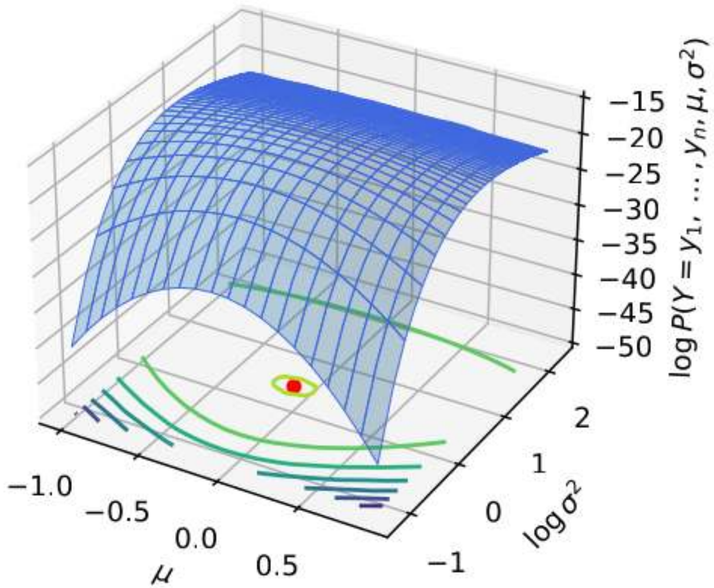
$y_1, \ldots, y_{10}$ drawn from

$\mathcal{N}(0, 1)$

	Data
y_1	1.624345
y_2	-0.611756
y_3	-0.528172
y_4	-1.072969
y_5	0.865408
y_6	-2.301539
y_7	1.744812
y_8	-0.761207
y_9	0.319039
y_{10}	-0.249370

Not mentioned: optimisation



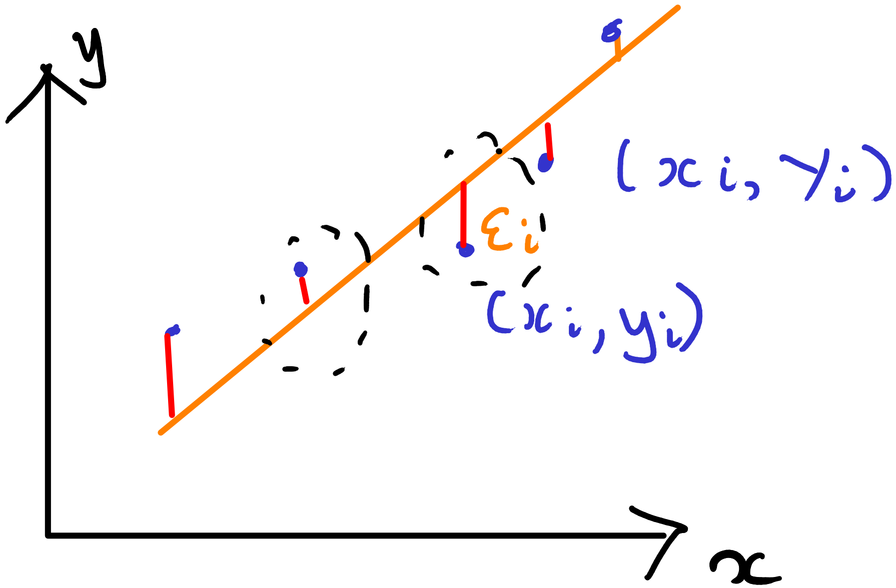


Application of max likelihood to linear regression

$$Y_i = \beta_0 + \beta_1 x_i + \underbrace{\varepsilon_i}_{\text{error term}}$$

$$\varepsilon_i \sim N(0, \sigma^2)$$

residual



OR

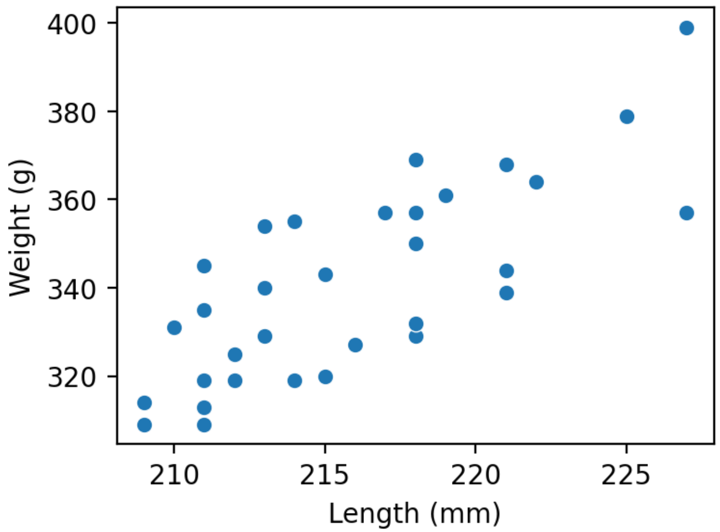
$$Y_i \sim N(\overbrace{\beta_0 + \beta_1 x_i}^{\mu}, \sigma^2)$$

$$\ln p(\underline{Y} = y_1, \dots, y_n ; x_1, \dots, x_n \mid \overbrace{\beta_0, \beta_1}^{\mu}, \sigma^2)$$

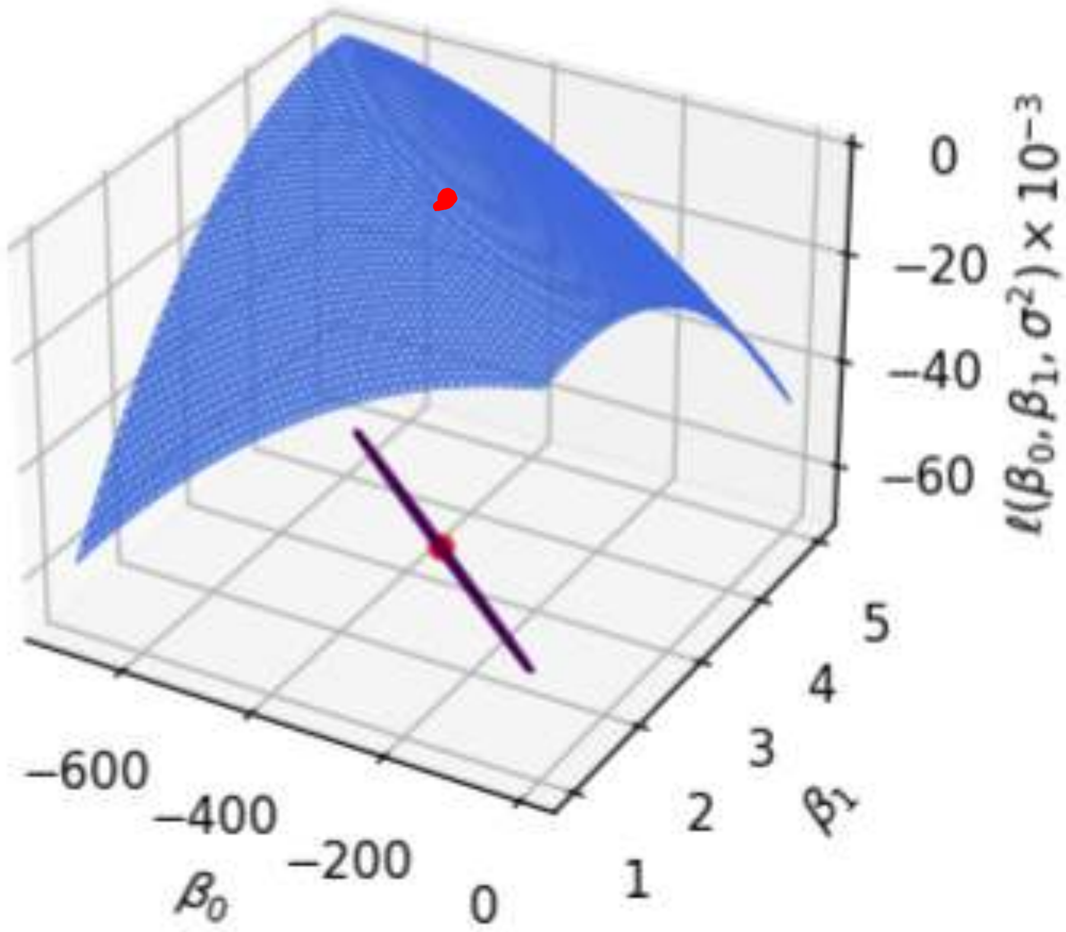
$$= \sum_{i=1}^{n} \left(-\frac{1}{2} \ln 2\pi\sigma^2 - \frac{1}{2} \frac{(y_i - \beta_0 - \beta_1 x_i)^2}{\sigma^2} \right)$$

Log likelihood of coefficients

Peter Trimming, Wikimedia Commons, CC BY 2.0



Data from Wauters and Dhondt 1989



Understanding more regression output

```
results = smf.ols('Weight ~ Length', data=datf).fit()
results.summary()
```

OLS Regression Results

Dep. Variable:	Weight	**R-squared:**	0.597
Model:	OLS	**Adj. R-squared:**	0.583
Method:	Least Squares	**F-statistic:**	44.37
Date:	Sun, 10 Jan 2021	**Prob (F-statistic):**	2.24e-07
Time:	21:08:04	**Log-Likelihood:**	-129.18
No. Observations:	32	**AIC:**	262.4
Df Residuals:	30	**BIC:**	265.3
Df Model:	1		
Covariance Type:	nonrobust		

	coef	std err	t	P>|t|	[0.025	0.975]
Intercept	-382.7372	108.680	-3.522	0.001	-604.692	-160.783
Length	3.3515	0.503	6.661	0.000	2.324	4.379

Omnibus:	8.046	**Durbin-Watson:**	2.337
Prob(Omnibus):	0.018	**Jarque-Bera (JB):**	2.231
Skew:	0.092	**Prob(JB):**	0.328
Kurtosis:	1.720	**Cond. No.**	9.38e+03

Bigger better

Max log likelihood $\ln \hat{L}$

Akaike Information criterion

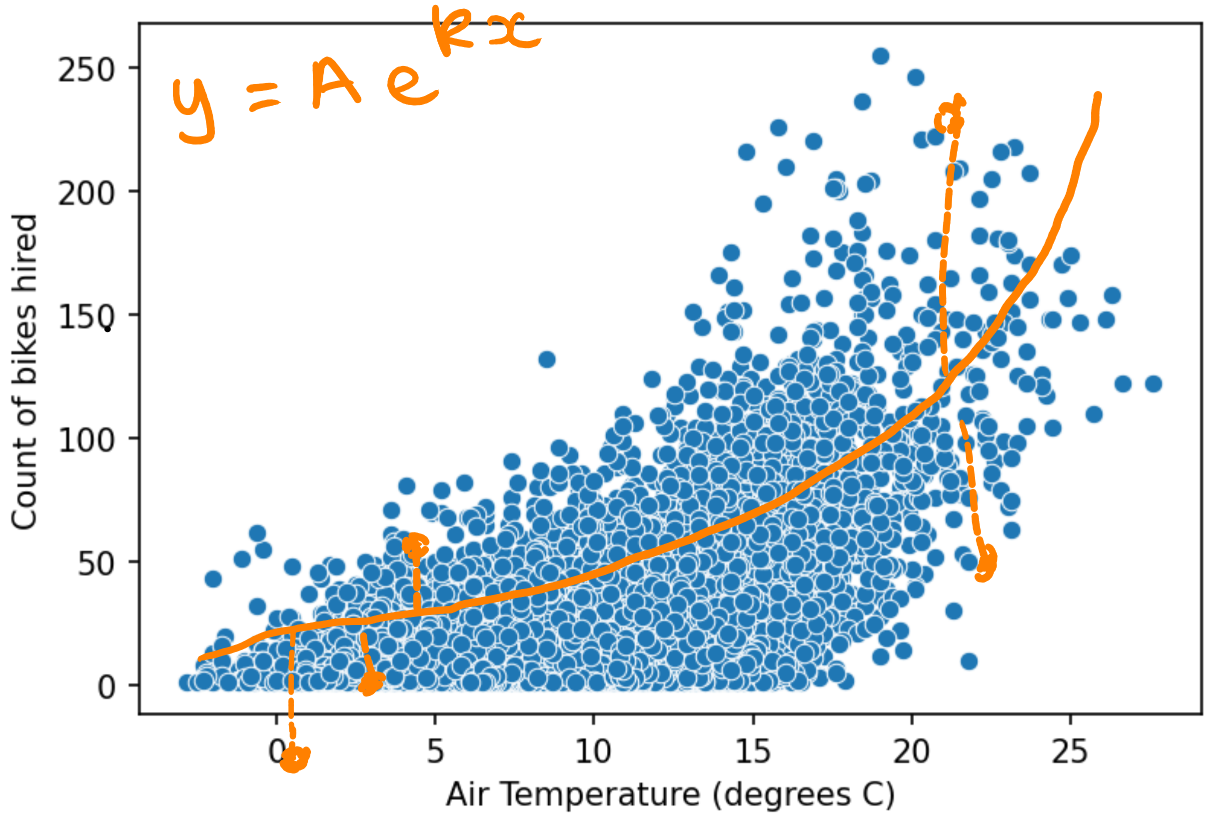
$$\underbrace{2k}_{\text{number of parameters}} - \underbrace{2\ln \hat{L}}_{\text{log likelihood}}$$

smaller better

We want to investigate the relationship between the number of bikes hired in an hour and the mean temperature during that hour

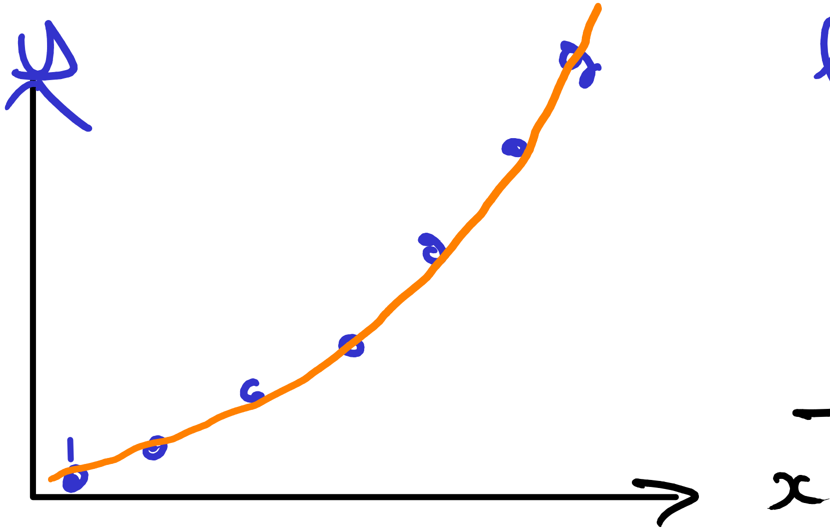
Is there a problem with using ordinary least squares linear regression to do this?

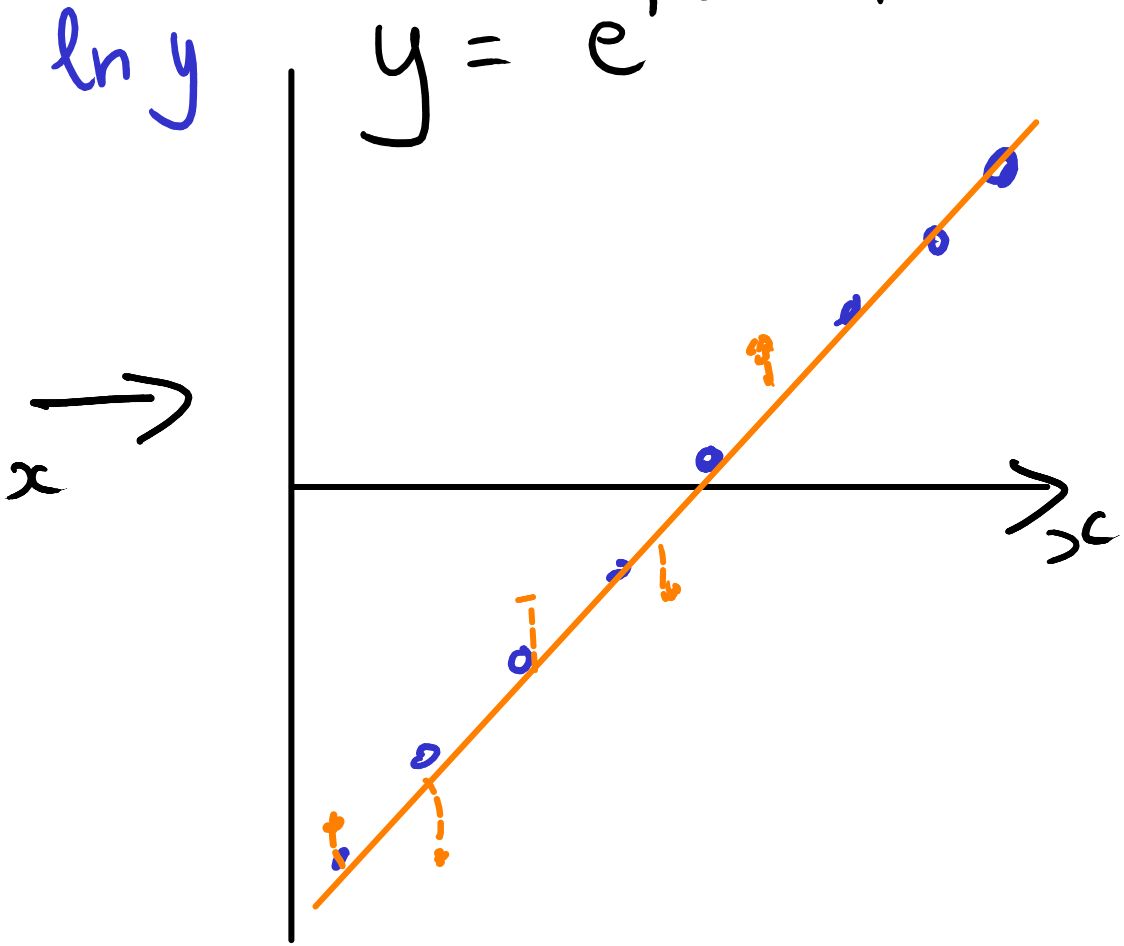
Are there any techniques described in the course so far that could fit the data?



$$\ln y = \beta_0 + \beta_1 x$$

$$y = e^{\beta_0 + \beta_1 x}$$





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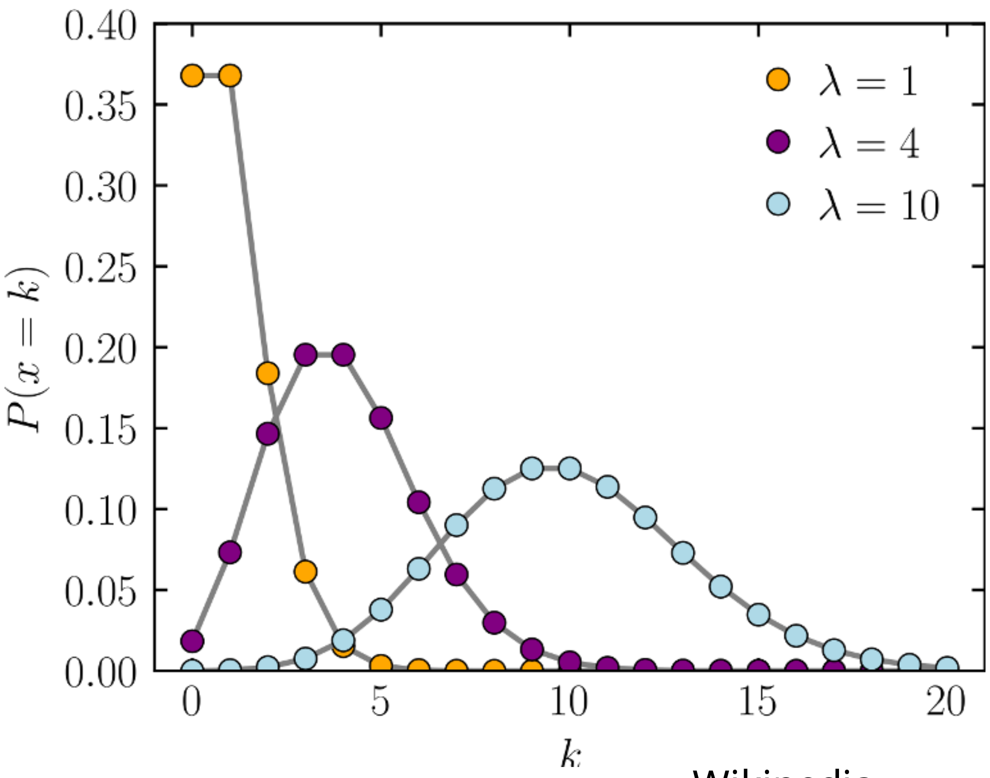
Max likelihood for models other than the normal

We don't have to assume the data is normally distributed.

E.g. Poisson distribution



Wikipedia

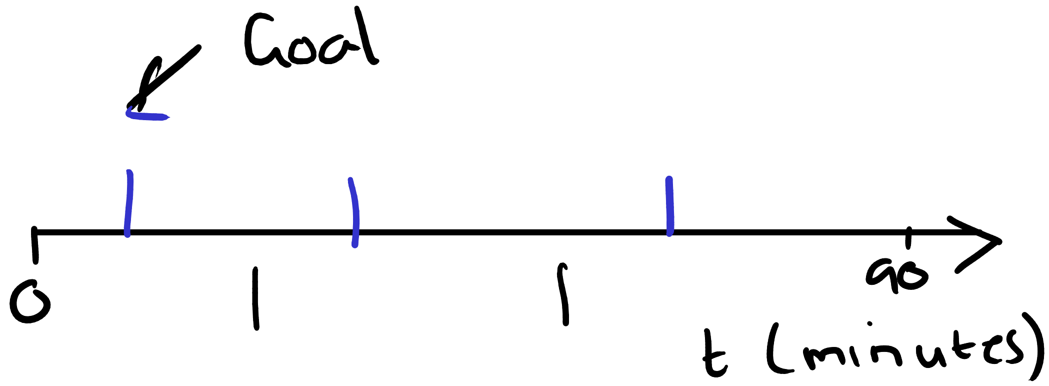
$$P(Y=k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

$k = 0, 1, 2, \ldots$

$E[Y] = \lambda = \mu$

$V[Y] = \lambda = \sigma^2$

E.g. Number of goals in World Cup football matches



Expected number of goals in a match $\lambda = 2.5$

$$P(Y=k) = \frac{2.5^k e^{-2.5}}{k!}$$

$$P(Y=0) = \frac{2.5^0 e^{-2.5}}{0!} = e^{-2.5} = 0.082$$

$$P(Y=1) = 0.205$$

$$P(Y=2) = 0.257$$

Number of deaths by horse kicks in the Prussian army

	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94
G	—	2	2	1	—	—	1	1	—	3	—	2	1	—	—	1	—	1	—	1
I	—	—	—	2	—	3	—	2	—	—	—	1	1	1	—	2	—	3	1	—
II	—	—	—	2	—	2	—	—	1	1	—	—	2	1	1	—	—	2	—	—
III	—	—	—	1	1	1	2	—	2	—	—	—	1	—	1	2	1	—	—	—
IV	—	1	—	1	1	1	1	—	—	—	—	1	—	—	—	—	1	1	—	—
V	—	—	—	—	2	1	—	—	1	—	—	1	—	1	1	1	1	1	1	—
VI	—	—	1	—	2	—	—	1	2	—	1	1	3	1	1	1	—	3	—	—
VII	1	—	1	—	—	—	1	—	1	1	—	—	2	—	—	2	1	—	2	—
VIII	1	—	—	—	1	—	—	1	—	—	—	—	1	—	—	—	1	1	—	1
IX	—	—	—	—	—	2	1	1	1	—	2	1	1	—	1	2	—	1	—	—
X	—	—	1	1	—	1	—	2	—	2	—	—	—	—	2	1	3	—	1	1
XI	—	—	—	—	2	4	—	1	3	—	1	1	1	1	2	1	3	1	3	1
XIV	1	1	2	1	1	3	—	4	—	1	—	3	2	1	—	2	1	1	—	—
XV	—	1	—	—	—	—	—	1	—	1	1	—	—	—	2	2	—	—	—	—

Bortkewitsch 1898

$$\underline{Y} = (0, 2, 2, 0, \ldots)$$

$y_1 \quad y_2 \quad \ldots \quad y_{280}$

$$n_k = \sum_{i=1}^{n} I(y_i = k)$$

k	n_k
0	144
1	91
2	32
3	11
4	2

Log likelihood calculation of Poisson distribution

Log likelihood $l = \ln P(\underline{Y} = y_1, \dots, y_n \mid \lambda)$

$$= \sum_{i=1}^{n} \ln P(Y = y_i)$$

$$= \sum_{i=1}^{n} \ln \frac{\lambda^{y_i} e^{-\lambda}}{y_i!}$$

$$= \sum_{i=1}^{n} \left(y_i \ln \lambda + (-\lambda) - \ln y_i! \right)$$

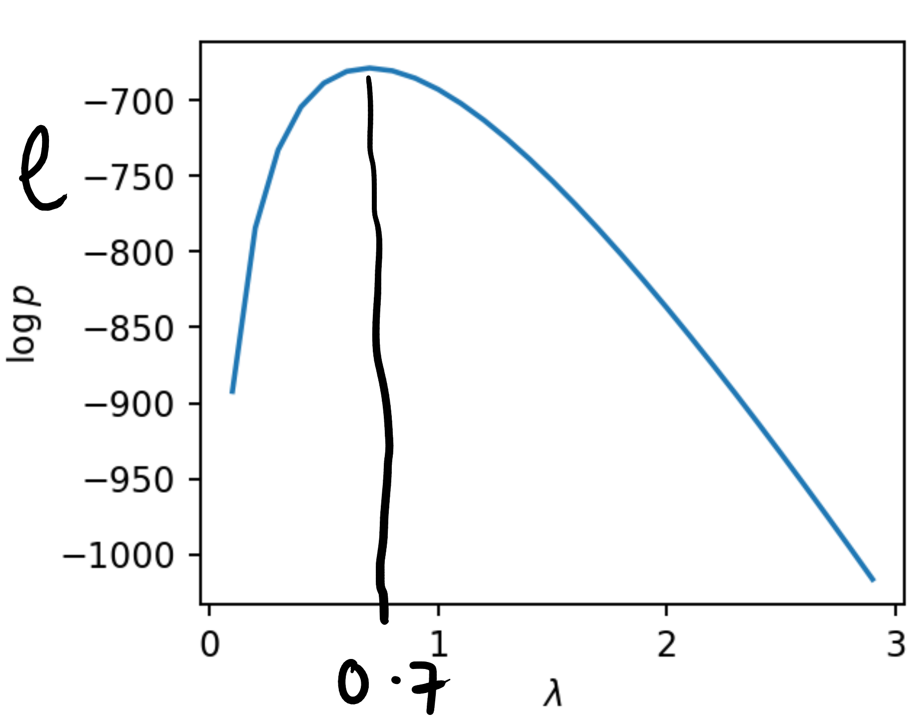
$$l(\lambda) = \ln \lambda \sum_{i=1}^{n} y_i - n\lambda - \sum_{i=1}^{n} \ln y_i!$$

$$\ell = \ln P(\underline{Y} = y_1, \ldots, y_n) = \ln \lambda \sum_{i=1}^{n} y_i - n\lambda - \sum_{i=1}^{n} \ln y_i!$$

$$\frac{d\ell}{d\lambda} = 0$$

$$\vdots$$

$$\Rightarrow \lambda_{MLE} = \frac{1}{n} \sum_{i=1}^{n} y_i$$



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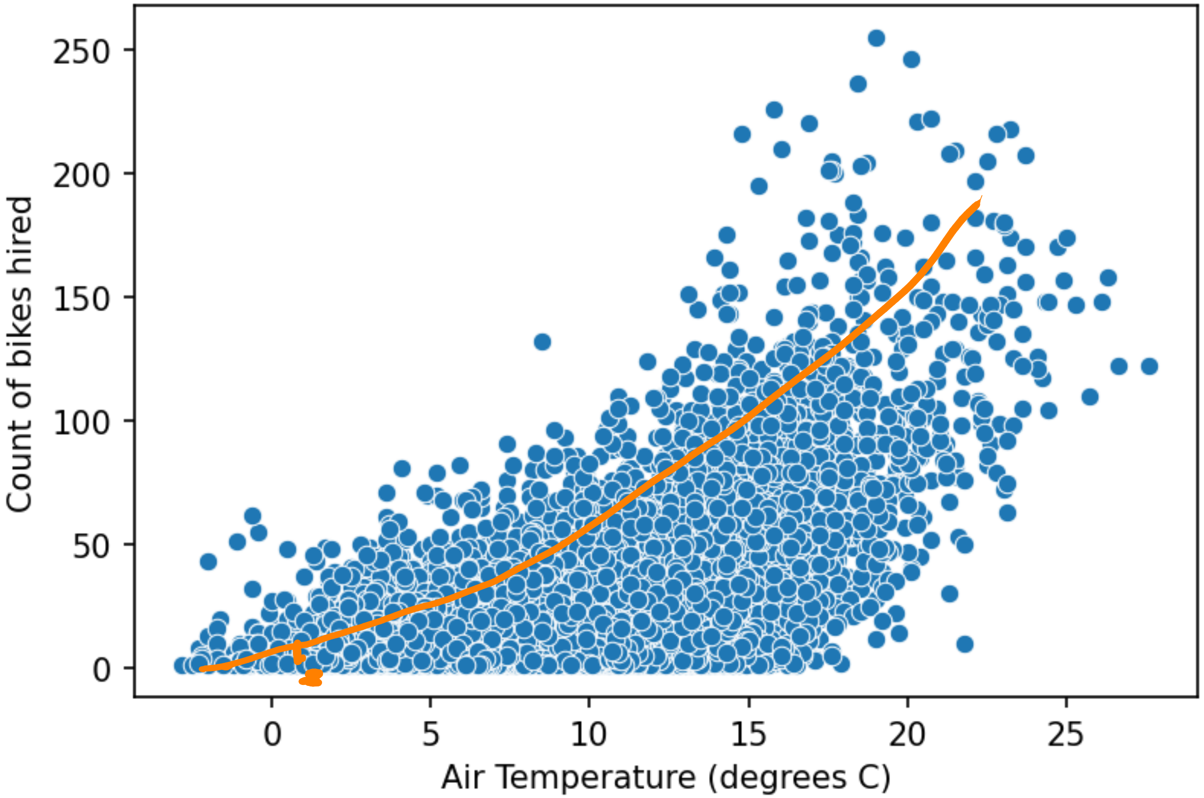
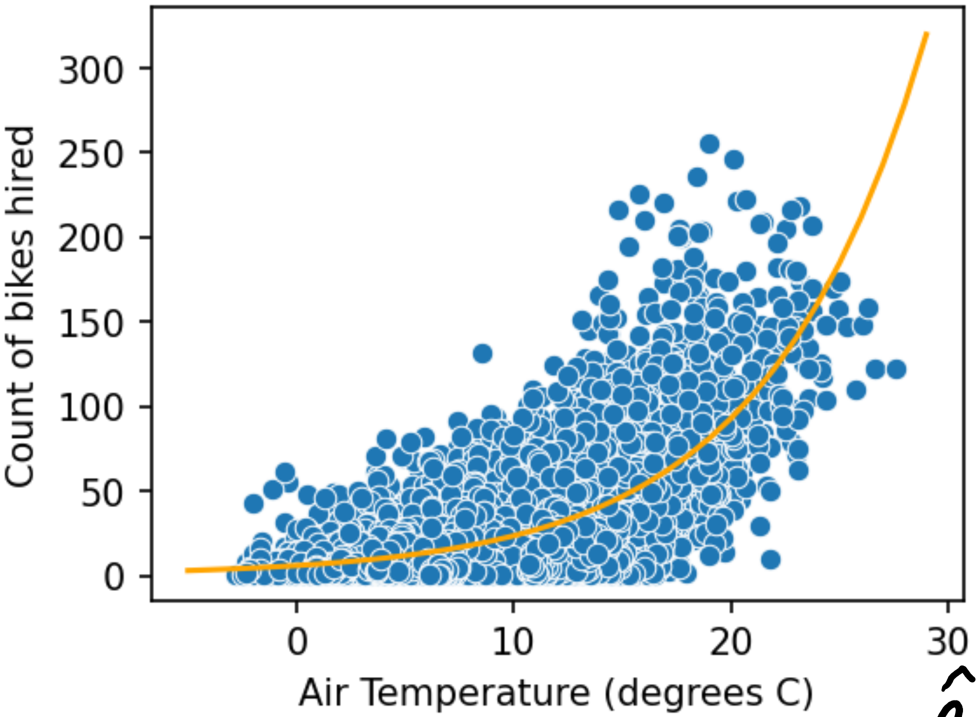
Poisson regression



$$Y_i \sim \text{Poisson}\left(\underbrace{e^{\beta_0 + \beta_1 x_i}}_{\lambda}\right)$$

$$\ln \lambda = \beta_0 + \beta_1 x$$

Results with statsmodels GLM



Generalized Linear Model Regression Results

Dep. Variable:	count	**No. Observations:**	8301
Model:	GLM	**Df Residuals:**	8299
Model Family:	Poisson	**Df Model:**	1
Link Function:	Log	**Scale:**	1.0000
Method:	IRLS	**Log-Likelihood:**	-84533.
Date:	Wed, 01 Mar 2023	**Deviance:**	1.3111e+05
Time:	06:46:41	**Pearson chi2:**	1.40e+05
No. Iterations:	5	**Pseudo R-squ. (CS):**	1.000
Covariance Type:	nonrobust		

	coef	**std err**	**z**	**P>|z|**	**[0.025**	**0.975]**
const	1.7861	0.006	304.092	0.000	1.775	1.798
air_temperature	0.1373	0.000	323.057	0.000	0.136	0.138

$\hat{\beta}_0$

$\hat{\beta}_1$

$$\ln \lambda = \hat{\beta}_0 + \hat{\beta}_1 x$$

$$\lambda = e^{\hat{\beta}_0 + \hat{\beta}_1 x} = e^{\hat{\beta}_0} \underbrace{e^{\hat{\beta}_1 x}}$$

$$e^{0.1373} = 1.14$$

Poisson regression

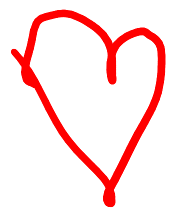
$$l = \ln P(\underline{Y} = y_1, \dots, y_n)$$

$$l(\beta_0, \beta_1) = \sum_{i=1}^{n} (\beta_0 + \beta_1 x_i) y_i - \sum_{i=1}^{n} e^{\beta_0 + \beta_1 x_i} - \sum_{i=1}^{n} \ln y_i!$$

Optimise β_0 & β_1

To my Valentine, Poisson Regression

Roses are red



Violets are blue

Some things aren't normal

and nor are you

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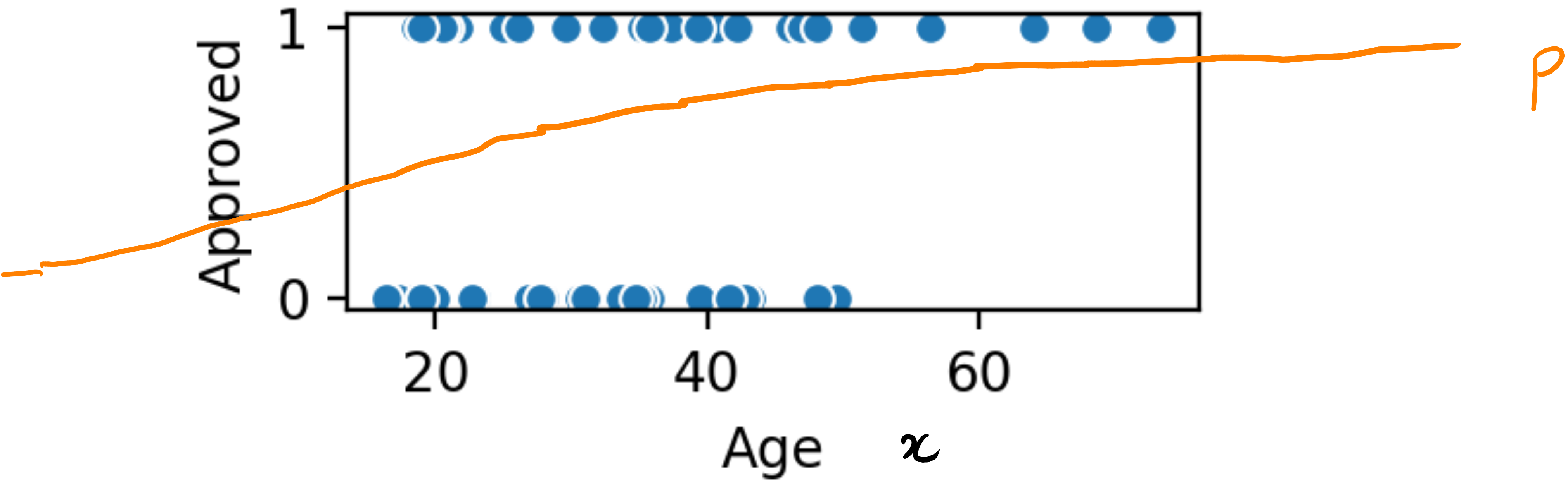
Excercise



What distribution would we use to model the data here?

Bernoulli - Parameter p

How would the parameter of that distribution depend on x (Age)?

Logistic function

$p = \text{Logistic}(\beta_0 + \beta_1 x)$

Generalised linear models (GLMs)

	Distribution	Link function
linear regression	Normal	$\mu = \beta_0 + \beta_1 x \quad , \sigma^2$
Poisson regression	Poisson	$\ln \lambda = \beta_0 + \beta_1 x$
logistic regression	Bernoulli	$\ln \dfrac{p}{1-p} = \beta_0 + \beta_1 x$

Link functions

Expected value $\mu = E(Y|x)$ of a Bernoulli dist is p

" " $\mu = E(Y|x)$ " " Poisson dist is λ

In general the link function is denoted $g(\mu)$ where $\mu = E(Y|x)$ for that distribution:

$$g(\mu) = \beta_0 + \beta_1 x$$

To make predictions, we invert the link function:

$$\mu = g^{-1}(\beta_0 + \beta_1 x)$$

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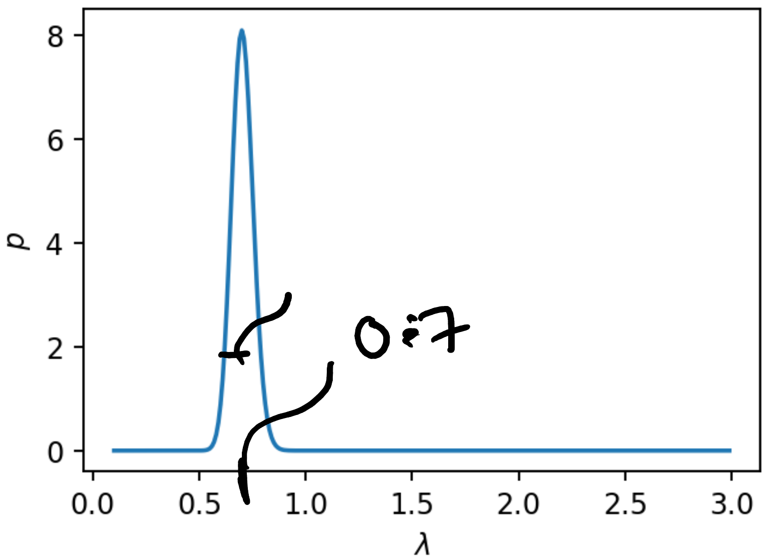
Max likelihood -> Bayesian Inference

Bayes Theorem:

$$\underbrace{P(\vartheta \mid \underline{Y} = y)}_{\text{Posterior}} = \frac{\overbrace{P(\underline{Y} = y \mid \vartheta)}^{\text{Likelihood}} \, \overbrace{p(\vartheta)}^{\text{Prior}}}{\underbrace{P(Y = y)}_{\text{Evidence}}}$$

Horsekick posterior



$$P(Y = y) = \int_{-\infty}^{\infty} P(Y = y \mid \vartheta)\, p(\vartheta)\, d\vartheta$$

Summary

Motivated the probabalistic basis of inference using max likelihood

Important: think of what distribution should describe the data

Links to future courses:
- MLG (derivation of standard ML methods)
- ATML (new in 25-26: cutting-edge machine learning)
- MCI (Causal inference)