# Introduction to Algorithms and Data Structures

## Lecture 14: Graphs, BFS, DFS

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# Directed and Undirected Graphs

- A *graph* is a mathematical structure consisting of a set of *vertices* and a set of *edges* connecting the vertices.

  *Formally:* $G = (V, E)$, where $V$ is a set and $E \subseteq V \times V$.

- $G = (V, E)$ *undirected* if for all $v, w \in V$:

$$(v, w) \in E \iff (w, v) \in E.$$

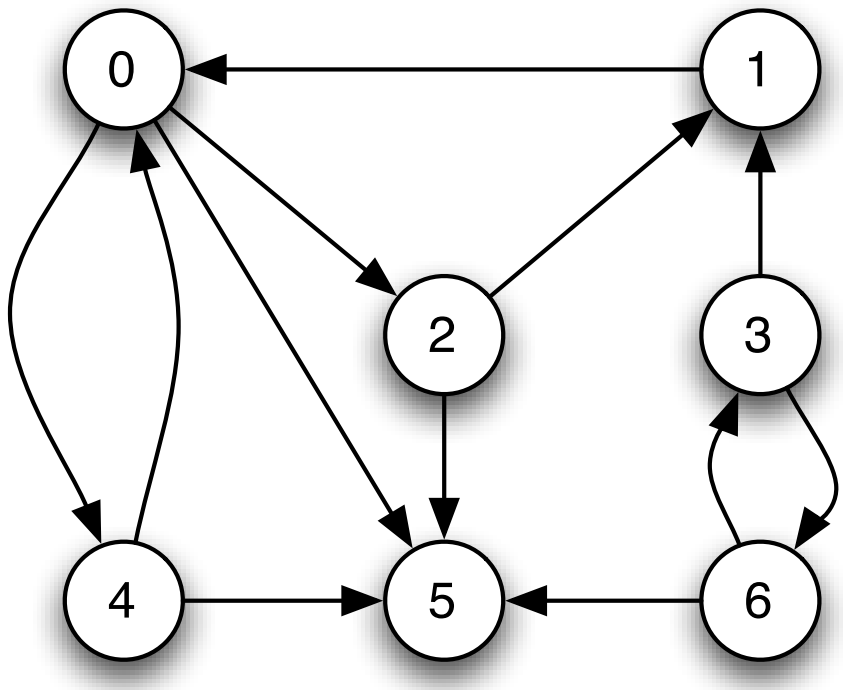
  Otherwise *directed*.

  Directed $\sim$ *arrows* (one-way)
  Undirected $\sim$ *lines* (two-way)

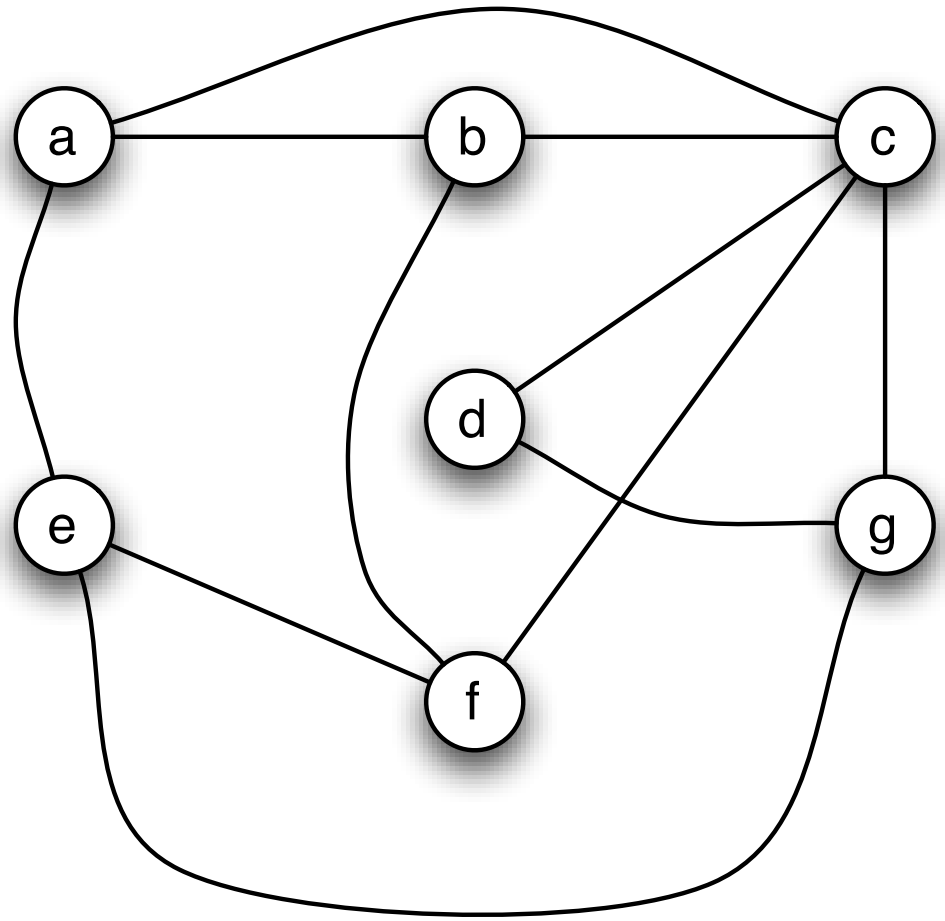
# A directed graph

$G = (V, E)$ with vertex set $V = \{0, 1, 2, 3, 4, 5, 6\}$ and edge set

$$\begin{aligned} E \quad = \quad & \{(0,2), (0,4), (0,5), (1,0), (2,1), (2,5), \\ & (3,1), (3,6), (4,0), (4,5), (6,3), (6,5)\}. \end{aligned}$$



# An undirected graph



# Examples of graphs in "real life"

*Road Maps*.
Edges represent streets and vertices represent crossings.

# Examples (cont'd)

*Computer Networks*.
Vertices represent computers and edges represent network connections (cables) between them.

*The World Wide Web*.
Vertices represent webpages, and edges represent hyperlinks.

# Adjacency matrices

Let $G = (V, E)$ be a graph with $n$ vertices, with vertices numbered $0, \ldots, n-1$.

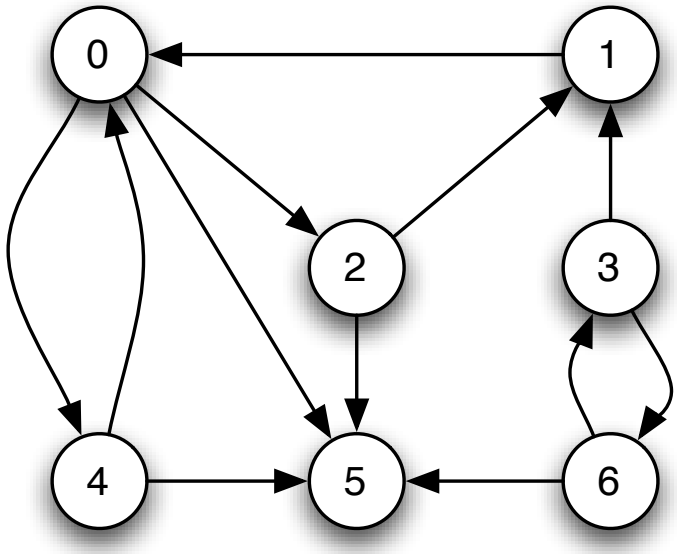
The *adjacency matrix* of $G$ is the $n \times n$ matrix $A = (a_{ij})_{0 \leq i,j \leq n-1}$ with

$$a_{ij} = \begin{cases} 1 & \text{if there is an edge from vertex } i \text{ to vertex } j \\ 0 & \text{otherwise.} \end{cases}$$

Python arrays are a bit strange to work with, being set up as "lists of lists". Alternatives are:

- Import `numpy`, and use their true 2D arrays
- Define a mapping $(i, j) \longrightarrow i * n + j$ (where $n$ is the number of vertices) and work with a $n * n$ or $n * (n+1)/2$ sized 1D array in python.

# Adjacency matrix (Example)

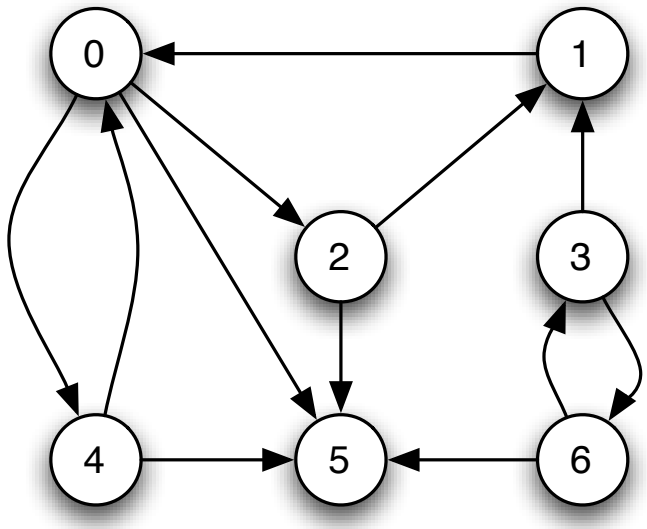


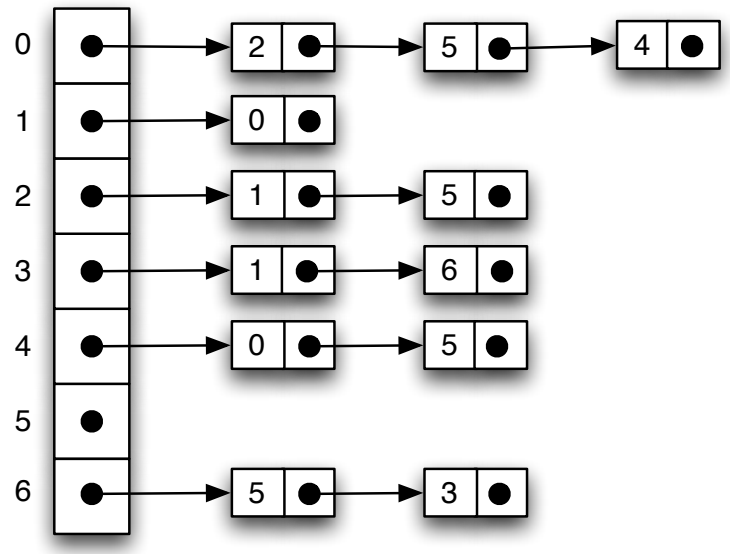

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

# Adjacency lists

Array with one entry for each vertex $v$, which is a list of all vertices adjacent to $v$.

## Example



# Lists or matrices?

Given: graph $G = (V, E)$, with $n = |V|$, $m = |E|$.
For $v \in V$, we write $in(v)$ for in-degree, $out(v)$ for out-degree.

Which data structure has faster (asymptotic) worst-case running-time, for *checking if $w$ is adjacent to $v$*, for a given pair of vertices?

1. Adjacency list is faster.
2. Adjacency matrix is faster.
3. Both have the same asymptotic worst-case running-time.
4. It depends.

# Finding neighbours

Given: graph $G = (V, E)$, with $n = |V|$, $m = |E|$.
For $v \in V$, we write $in(v)$ for in-degree, $out(v)$ for out-degree.

Which data structure has faster (asymptotic) worst-case running-time, for *visiting all vertices $w$ adjacent to $v$*, for a given vertex $v$?

1. Adjacency list is faster.
2. Adjacency matrix is faster.
3. Both have the same asymptotic worst-case running-time.
4. It depends.

# Adjacency Matrices vs Adjacency Lists

| | adjacency matrix | adjacency list |
|---|---|---|
| Space | $\Theta(n^2)$ | $\Theta(n+m)$ |
| Time to check if $w$ adjacent to $v$ | $\Theta(1)$ | $\Theta(out(v))$ |
| Time to visit all $w$ adjacent to $v$. | $\Theta(n)$ | $\Theta(out(v))$ |
| Time to visit all edges | $\Theta(n^2)$ | $\Theta(n+m)$ |

# Sparse and dense graphs

$G = (V, E)$ graph with $n$ vertices and $m$ edges

Observation: $m \leq n^2$

- $G$ *dense* if $m$ close to $n^2$
- $G$ *sparse* if $m$ much smaller than $n^2$

# Graph traversals

A *traversal* is a strategy for visiting all vertices of a graph.

*BFS = breadth-first search*

*DFS = depth-first search*

General strategy:

1. Let $v$ be an arbitrary vertex
2. Visit all vertices reachable from $v$
3. If there are vertices that have not been visited, let $v$ be such a vertex and go back to 2.

# BFS

Visit all vertices reachable from $v$ in the following order:

- $v$
- all neighbours of $v$
- all neighbours of neighbours of $v$ that have not been visited yet
- all neighbours of neighbours of neighbours of $v$ that have not been visited yet
- etc.

# BFS (using a Queue)

**Algorithm** bfs($G$)

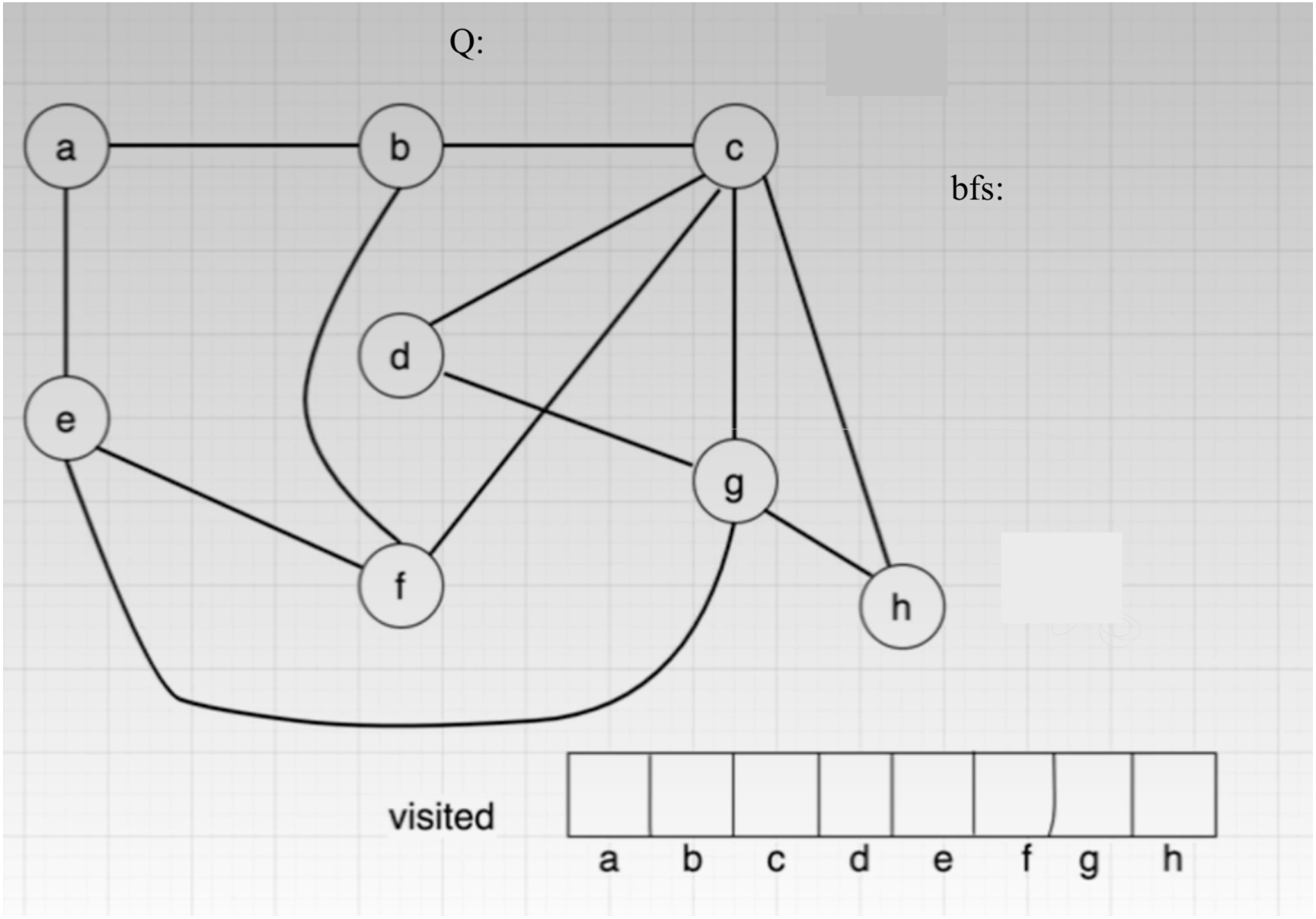
1. Initialise Boolean array $visited$, setting all entries to FALSE.
2. Initialise *Queue* $Q$
3. **for all** $v \in V$ **do**
4. **if** $visited[v] = \text{FALSE}$ **then**
5. bfsFromVertex($G, v$)

# BFS (using a Queue)

**Algorithm** bfsFromVertex($G, v$)

1. $visited[v] = \text{TRUE}$
2. $Q$.enqueue($v$)
3. **while not** $Q$.isEmpty() **do**
4. $\quad v \leftarrow Q$.dequeue()
5. $\quad$ **for all** $w$ adjacent to $v$ **do**
6. $\quad\quad$ **if** $visited[w] = \text{FALSE}$ **then**
7. $\quad\quad\quad visited[w] = \text{TRUE}$
8. $\quad\quad\quad Q$.enqueue($w$)

# BFS worked example



# Asymptotic running-time of BFS

Given a graph $G = (V, E)$ with $n = |V|$, $m = |E|$, what is the worst-case running time of BFS, in terms of $m, n$?

1. $\Theta(m + n)$
2. $\Theta(n^2)$
3. $\Theta(mn)$
4. Depends on the number of components.

# DFS

Visit all vertices reachable from $v$ in the following order:

- $v$
- some neighbor $w$ of $v$ that has not been visited yet
- some neighbor $x$ of $w$ that has not been visited yet
- etc., until the current vertex has no neighbour that has not been visited yet
- Backtrack to the first vertex that has a yet unvisited neighbour $v'$.
- Continue with $v'$, a neighbour, a neighbour of the neighbour, etc., backtrack, etc.

# Reading

Today's lecture:

- Graph representations in Section 22.1
- Breadth-first search Section 22.2

Tuesday next:

- Depth-first search Section 22.3
- Topological sort Section 22.4