Introduction to Algorithms and Data Structures

Lecture 15: DFS and graph structure

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DFS (using a stack)

Algorithm dfs(G)

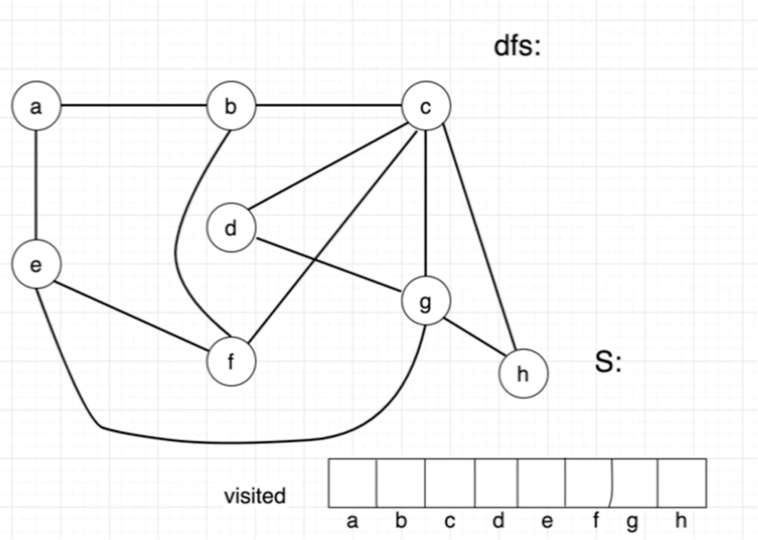
1. Initialise Boolean array $visited$, setting all to FALSE
2. Initialise *Stack* S
3. **for all** $v \in V$ **do**
4.   **if** $visited[v] = \text{FALSE}$ **then**
5.   dfsFromVertex(G, v)

DFS (using a stack)

Algorithm dfsFromVertex(G, v)

```
S.push(v)
while not S.isEmpty() do
    u ← S.pop()
    if visited[u] = FALSE then
        visited[u] ← TRUE
    for all w adjacent to u do
        if visited[w] = FALSE then
            S.push(w)
```

DFS worked example



Recursive DFS (no explicit Stack)

Algorithm dfs(G)

Initialise Boolean array $\mathit{visited}$, setting all entries to FALSE
for all $v \in V$ **do**
    if $\mathit{visited}[v] = \text{FALSE}$ **then**
        dfsFromVertex(G, v)

Algorithm dfsFromVertex(G, v)

$\mathit{visited}[v] \leftarrow \text{TRUE}$
for all w adjacent to v **do**
    if $\mathit{visited}[w] = \text{FALSE}$ **then**
        dfsFromVertex(G, w)

(We will have reversed prioritisation of the vertices adjacent to v, compared to the Stack version)

Analysis of Recursive DFS

Lemma

During dfs(G)*,* dfsFromVertex(G, v) *is invoked exactly once for each vertex* v*.*

Proof.

At least once:

- $\mathit{visited}[v]$ can only become TRUE when dfsFromVertex(G, v) is executed.
- If $\mathit{visited}[v]$ remains FALSE, dfsFromVertex(G, v) will eventually be called by line 4 of dfs(G).

At most once:

- First call of dfsFromVertex(G, v) sets $\mathit{visited}[v]$ to TRUE.
- After $\mathit{visited}[v]$ is TRUE, dfsFromVertex(G, v) is *never* called again.

□

("At most once" is also true for Stack dfs, but "at least once" is not. dfsFromVertex" is more to "start a component" in the Stack version)

Analysis of DFS (cont'd)

Lemma

For a directed graph, $\sum_{v \in V} \textit{out-degree}(v) = m$.
For an undirected graph, $\sum_{v \in V} \textit{deg}(v) = 2m$.

Proof.

Every edge is counted exactly once on both sides of the equation (for directed).

For the undirected case, every edge is counted twice on the lhs. □

Analysis of recursive DFS

$G = (V, E)$ graph with n vertices and m edges

Algorithm dfs(G)

1. Initialise Boolean array *visited*, setting all to FALSE
2. **for all** $v \in V$ **do**
3.   **if** *visited*$[v] =$ FALSE **then**
4.   dfsFromVertex(G, v)

- dfs(G): Ignoring calls to dfsFromVertex, total time $\Theta(n)$
- dfsFromVertex(v) is called at most once for every vertex v, and does $\Theta(\text{out-degree}(v))$ work, excluding recursive calls.

Overall time:

$$
\begin{aligned}
T(n, m) &= \Theta(n) + \textstyle\sum_{v \in V} \Theta(\text{out-degree}(v)) \\
&= \Theta\Big(n + \textstyle\sum_{v \in V} \text{out-degree}(v)\Big) \\
&= \Theta(n + m)
\end{aligned}
$$

Adjacency List or Adjacency Matrix?

We said each call to dfsFromVertex(v) takes $\Theta(\text{out-degree}(v))$ time (excluding recursive calls).

Algorithm dfsFromVertex(G, v)

1. $\mathit{visited}[v] \leftarrow \text{TRUE}$
2. **for all** w adjacent to v **do**
3.   **if** $\mathit{visited}[w] = \text{FALSE}$ **then**
4.   dfsFromVertex(G, w)

If we are iterating over "all w adjacent to v" in $\Theta(\text{out-degree}(v))$ time, then we must be using an Adjacency list structure.

Analysis of Stack DFS

Compare the two dfsFromVertex(G, v) methods:

Algorithm dfsFromVertex(G, v)

```
visited[v] ← TRUE
for all w adjacent to v do
    if visited[w] = FALSE then
        dfsFromVertex(G, w)
```

Algorithm dfsFromVertex(G, v)

```
S.push(v)
while not S.isEmpty() do
    u ← S.pop()
    if visited[u] = FALSE then
        visited[u] ← TRUE
    for all w adjacent to u do
        if visited[w] = FALSE then
            S.push(w)
```

$\Big|\ visited[w] \leftarrow \text{TRUE}\ \Big|\ \leftrightarrow\ \Big|\ u \leftarrow S.\text{pop}();\ visited[u] \leftarrow \text{TRUE};\ \Big|$

Recursive: marks v as "visited", *then* calls dfsFromVertex for unvisited adjacent vertices

Iterative: "pops" v off top to mark as "visited" and explore/push adjacent vertices.

However, the number of Stack operations for v is bounded in terms of *the number of edges into* $v \Rightarrow$ the overall runtime for our original dfs is still $\Theta(n + m)$.

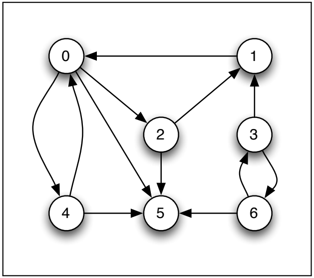
DFS Forests

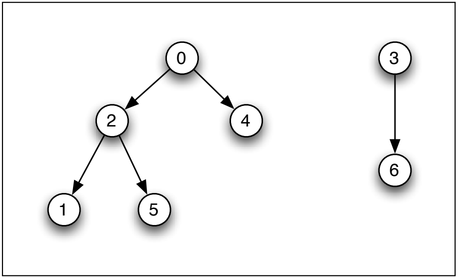
A DFS traversing a graph builds up a forest whose vertices are all vertices of the graph and whose edges are all vertices traversed during the DFS.

Definition

A vertex w is a *child* of a vertex v in the DFS forest if dfsFromVertex(G, w) is called from dfsFromVertex(G, v).

DFS Forests Example





On directed graphs, the connected components (trees) might vary depending on the order in which we consider vertices at the top-level of dfs.

Topological Sorting

Example:
10 tasks to be carried out. Some of them depend on others.

- Task 0 must be completed before Task 1 can be started.
- Task 1 and Task 2 must be done before Task 3 can start.
- Task 4 must be done before Task 0 or Task 2 can start.
- Task 5 must be done before Task 0 or Task 4 can start.
- Task 6 must be done before Task 4, 5 or 7 can start.
- Task 7 must be done before Task 0 or Task 9 can start.
- Task 8 must be done before Task 7 or Task 9 can start.
- Task 9 must be done before Task 2 or Task 3 can start.

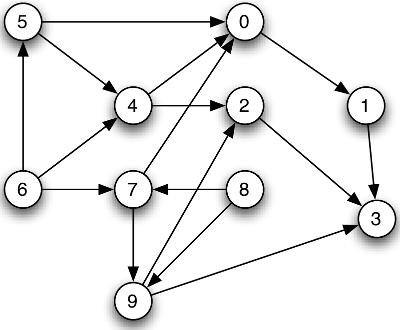
Topological order

Definition

Let $G = (V, E)$ be a directed graph. A *topological order* of G is a total order $\prec$ of the vertex set V such that for all edges $(v, w) \in E$ we have $v \prec w$.

(in some fields this is called a *linear extension*)

Tasks as a (directed) graph



Does this graph have a topological order?

Yes. One topological sort is:

$$8 \prec 6 \prec 7 \prec 9 \prec 5 \prec 4 \prec 2 \prec 0 \prec 1 \prec 3.$$

Topological order (cont'd)

A digraph that has a cycle does not have a topological order.

Definition

A *DAG* (**d**irected **a**cyclic **g**raph) is a digraph without cycles.

Theorem

A digraph has a topological order if and only if it is a DAG.

Classification of vertices during recursive DFS

$G = (V, E)$ graph, $v \in V$. Consider dfs(G).

- v is finished if dfsFromVertex(G, v) has been completed.

Vertices can be in the following states:

- not yet visited (let us call a vertex in this state *white*),
- visited, but not yet finished (*grey*).
- finished (*black*).

(note these colours are *explicitly* marked in version of DFS by [CLRS] 22.3)

Classification of vertices during recursive DFS (cont'd)

Lemma

Let G be a graph and v a vertex of G. Consider the moment during the execution of dfs(G) *when* dfsFromVertex(G, v) *is started.*
Then for all vertices w we have:

1. *If w is* white *and reachable from v, then w will be* black *before v.*
2. *If w is* grey*, then v is reachable from w.*

Topological sorting

$G = (V, E)$ digraph. Define order on V as follows:

$$v \prec w \iff w \text{ becomes black before } v.$$

Theorem

If G is a DAG then $\prec$ is a topological order.

Proof.

Suppose $(v, w) \in E$. Consider dfsFromVertex(G, v).

- If w is already *black*, then $v \prec w$ (and this is what we want).
- If w is *white*, then by Lemma part 1., w will be *black* before v. Thus $v \prec w$.
- If w is *grey*, then by Lemma part 2. v is reachable from w. So there is a path p from w to v. Path p and edge (v, w) together form a cycle. **Contradiction!** (G is acyclic ...)

□

Topological sorting implemented

Algorithm topSort(G)

1. Initialise array $state$
   by setting all entries to $white$.
2. Initialise linked list L
3. **for all** $v \in V$ **do**
4.   **if** $state[v] = white$ **then**
5.   sortFromVertex(G, v)
6. **print** L

Topological sorting implemented

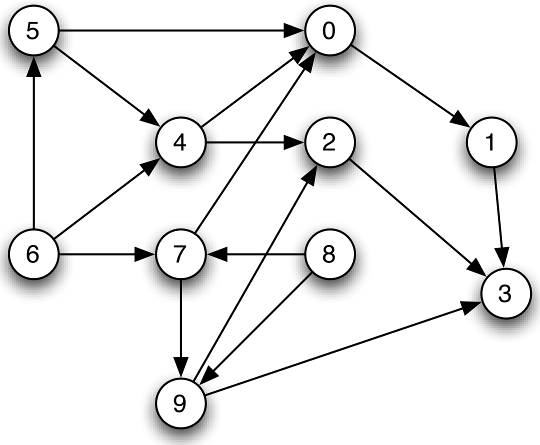
Algorithm sortFromVertex(G, v)

```
state[v] ← grey
for all w adjacent to v do
    if state[w] = white then
        sortFromVertex(G, w)
    else if state[w] = grey then
        print "G has a cycle"
        halt
state[v] ← black
L.insertFirst(v)
```

Difference from dfs itself - the order the vertices get added to the list.

Running-time is again $\Theta(n + m)$

Example



Use the algorithm topSort to compute a topological sort of this graph.

Connected components of an undirected graph

$G = (V, E)$ undirected graph

Definition

- A subset C of V is *connected* if for all $v, w \in C$ there is a path from v to w (if G is directed, say *strongly connected*).
- A *connected component* of G is a maximum connected subset C of V. (no connected subset C' of V strictly contains C.
- G is *connected* if it only has one connected component, that is, if for all vertices v, w there is a path from v to w.

Connected components - undirected (cont'd)

- Each vertex of an undirected graph is contained in exactly one connected component.
- For each vertex v of an undirected graph, the connected component that contains v is precisely the set of all vertices that are reachable from v.

For an undirected graph G, dfsFromVertex(G, v) visits exactly the vertices in the connected component of v.

And the same is true for bfsFromVertex(G, v) (either will do!)

Connected components - undirected (cont'd)

Algorithm connComp(G)

1. Initialise Boolean array $visited$
   by setting all entries to FALSE
2. **for all** $v \in V$ **do**
3.   **if** $visited[v] = \text{FALSE}$ **then**
4.   **print** "New Component"
5.   ccFromVertex(G, v)

Algorithm ccFromVertex(G, v)

1. $visited[v] \leftarrow \text{TRUE}$
2. **print** v
3. **for all** w adjacent to v **do**
4.   **if** $visited[w] = \text{FALSE}$ **then**
5.   ccFromVertex(G, w)

Reading

From [CLRS:

- Depth-first search - Section 22.3
- Computing topological sort - Section 22.4

From "Algorithms Illuminated":

- sections 8.3, 8.4, 8.5

Hope you get a break over the holidays!

And "see" you in 2025.