Informatics 2D: Reasoning and Agents

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Lecture 23: Probabilistic Reasoning with Bayesian Networks

Where are we?

Last time ...

- Using JPD tables for probabilistic inference
- Concepts of absolute and conditional independence
- Bayes' rule

Today ...

- **Probabilistic Reasoning with Bayesian Networks**

Representing knowledge in an uncertain domain

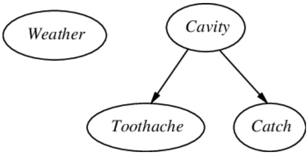
- Full joint probability distributions can become intractably large very quickly
- Conditional independence helps to reduce the number of probabilities required to specify the JPD
- Now we will introduce **Bayesian networks** (BNs) to systematically describe dependencies between random variables
- Roughly speaking, BNs are graphs that connect nodes representing variables with each other whenever they depend on each other

Bayesian networks

- A BN is a directed acyclic graph (DAG) with nodes annotated with probability information
- The nodes represent random variables (discrete/continuous)
- Links connect nodes. If there is an arrow from X to Y, we call X a **parent** of Y
- Each node X_i has a conditional probability distribution (CPD) attached to it
- The CPD describes how X_i depends on its parents, i.e. its entries describe $\mathbf{P}(X_i|Parents(X_i))$

Bayesian networks

- Topology of graphs describes conditional independence relationships
- Intuitively, links describe **direct effects** of variables on each other in the domain
- Assumption: anything that is not directly connected does not directly depend on each other
- In previous dentist/weather example:



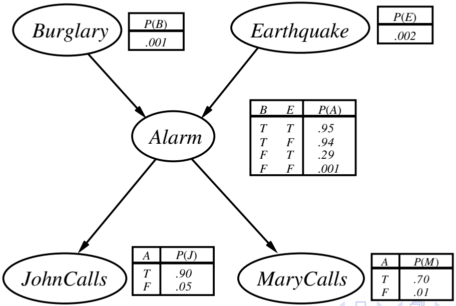
Arcs and Independence

Each variable is conditionally independent of its non-descendants, given its parents.

$$\text{If } X \notin Parents^{*}(Y)\text{, then}$$
$$\mathrm{P}(X|Parents(X), Y) = \mathrm{P}(X|Parents(X))$$

Example

- New burglar alarm has been fitted, fairly reliable but sometimes reacts to earthquakes
- Neighbours John and Mary promise to call when they hear alarm
- John sometimes mistakes phone for alarm, and Mary listens to loud music and sometimes doesn't hear alarm



Example – things to note

- No perception of earthquake by John or Mary
- No explicit modelling of phone ring confusing John, or of Mary's loud music
  (summarised in uncertainty regarding their reaction)
- Actually this uncertainty summarises any kind of failure
  - almost impossible to enumerate all possible causes,
  - and we don't have estimates for their probabilities anyway
- Each row in CPTs contains a **conditioning case**,
  one row for each possible combination of values of the parents.
- We often omit $P(\neg x_i|Parents(X_i))$ from CPT for node X_i
  (computes as $1 - P(x_i|Parents(X_i))$)
- $\mathrm{P}(J|M,A,B,E) = \mathrm{P}(J|A)$ and $\mathrm{P}(M|J,A,B,E) = \mathrm{P}(M|A)$

The semantics of Bayesian Networks

- Two views:
  - BN as representation of JPD (useful for constructing BNs)
  - BN as collection of conditional independence statements (useful for designing inference procedures)
- Every entry $P(X_1 = x_1 \wedge \ldots \wedge X_n = x_n)$ in the JPD can be calculated from a BN (abbreviate by $P(x_1, \ldots, x_n)$)
- $P(x_1, \ldots, x_n) = \prod_{i=1}^{n} P(x_i | parents(X_i))$
- Example:

$$\begin{aligned} &P(j \wedge m \wedge a \wedge \neg b \wedge \neg e) \\ &\quad = P(j|a)P(m|a)P(a|\neg b \wedge \neg e)P(\neg b)P(\neg e) \\ &\quad = 0.9 \times 0.7 \times 0.001 \times 0.999 \times 0.998 = 0.00062 \end{aligned}$$

- As before, this can be used to answer any query

A method for constructing BNs

- Recall product rule for n variables:

$$P(x_1,\ldots,x_n) = P(x_n|x_{n-1},\ldots,x_1)P(x_{n-1},\ldots,x_1)$$

- Repeated application of this yields the so-called **chain rule**:

$$\begin{aligned} P(x_1,\ldots,x_n) &= P(x_n|x_{n-1},\ldots,x_1)P(x_{n-1}|x_{n-2},\ldots,x_1)\cdots P(x_2|x_1)P(x_1) \\ &= \prod_{i=1}^{n} P(x_i|x_{i-1},\ldots,x_1) \end{aligned}$$

- With this we obtain $\mathrm{P}(X_i|X_{i-1},\ldots,X_1) = \mathrm{P}(X_i|Parents(X_i))$ as long as $Parents(X_i) \subseteq \{X_{i-1},\ldots,X_1\}$ (this can be ensured by labelling nodes appropriately)
- For example, it is reasonable to assume that

$$\mathrm{P}(MaryCalls|JohnCalls, Alarm, Earthquake, Burglary) = \mathrm{P}(MaryCalls|Alarm)$$

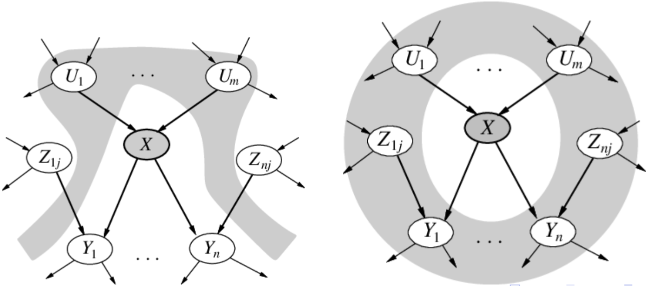
Compactness and node ordering

- BNs examples of **locally structured** (**sparse**) systems: subcomponents only interact with small number of other components
- E.g. if 30 nodes and every node depends on 5 nodes, BN will have $30 \times 2^5 = 960$ probabilities stored in the CPDs, while JPD would have $2^{30} \approx 1000^3$ entries
- But remember that this is based on designer's independence assumptions!
- Also not trivial to determine good BN structure:

  Add "root causes" first, then variables they influence, and so on, until we reach "leaves" which have no influence on other variables

Conditional independence relations in BNs

- Have provided "numerical" semantics, but can also look at (equivalent) "topological" semantics, namely:
  1. A node is conditionally independent of its **non-descendants**, given its parents
  2. A node is conditionally independent of all other nodes, given its parents, children and children's parents, i.e. its **Markov blanket**



Efficient representation of conditional distributions

- Even the 2^k (k parents) conditioning cases that have to be provided require a great deal of experience and knowledge of the domain
- Arbitrary relationships are unlikely, often describable by **canonical distributions** that fit some standard pattern
- By specifying pattern by a few parameters we can save a lot of space!
- Simplest case: **deterministic node** that can be directly inferred from values of parents
- For example, logical or mathematical functions

Noisy-OR relationships

Generalisation of logical OR

- Any cause *can* make effect true, but won't *necessarily* (effect **inhibited**; $P(effect|cause) < 1$)
- Assumes all causes are listed (**leak node** can be used to cater for "miscellaneous" unlisted causes)
- Also assumes inhibitions are mutually conditionally independent
  - Whatever inhibits C_1 from making E true is independent of what inhibits C_2 from making E true.
- So E is *false* only if each of its *true* parents are inhibited and we can compute this likelihood from product of probabilities for each individual cause inhibiting E.
- How does this help?

Example of Noisy-OR

- *Fever* is caused by *Cold*, *Flu* or *Malaria* and that's all (!!)
- Inhibitions of *Cold*, *Flu* and *Malaira* are mutually conditionally independent
- Likelihood that *Cold* is inhibited from causing *Fever* is $P(\neg fever|cold, \neg flu, \neg malaria)$ (similarly for other causes)
- Individual inhibition probabilities:

$$P(\neg fever|cold, \neg flu, \neg malaria) = 0.6$$
$$P(\neg fever|\neg cold, flu, \neg malaria) = 0.2$$
$$P(\neg fever|\neg cold, \neg flu, malaria) = 0.1$$

- Inhibitions mutually independent, so:

$$P(\neg fever|cold, flu, \neg malaria) = P(\neg fever|cold, \neg flu, \neg malaria)P(\neg fever|\neg cold, flu, \neg malaria)$$

Noisy-OR relationships

- We can construct entire CPT from this information

$Cold$	Flu	$Malaria$	$P(Fever)$	$P(\neg Fever)$
F	F	F	0.0	1.0
F	F	T	0.9	**0.1**
F	T	F	0.8	**0.2**
F	T	T	0.98	$0.02 = 0.2 \times 0.1$
T	F	F	0.4	**0.6**
T	F	T	0.94	$0.06 = 0.6 \times 0.1$
T	T	F	0.88	$0.12 = 0.6 \times 0.2$
T	T	T	0.988	$0.012 = 0.6 \times 0.2 \times 0.1$

- Encodes CPT with k instead of 2^k values!

BNs with continuous variables

- Often variables range over continuous domains
- **Discretisation** one possible solution but often leads to inaccuracy or requires a lot of discrete values
- Other solution: use of standard families of probability distributions specified in terms of a few parameters
- Example: normal/Gaussian distribution $N(\mu, \sigma^2)(x)$ defined in terms of mean μ and variance σ^2 (needs just two parameters)
- **Hybrid Bayesian Networks** use mixture of discrete and continuous variables (special methods to deal with links between different types – not discussed here)

Summary

- Introduced Bayesian Networks as a structured way of reasoning under uncertainty using probabilities and independence
- Defined their semantics in terms of JPD representation, and conditional independence statements
- Gave numerical and topological interpretation of semantics
- Talked about issues of efficient representation of CPTs
- Discussed continuous variables and hybrid networks
- Next time: **Exact Inference in Bayesian Networks**