Informatics 2D: Reasoning and Agents

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Lecture 27a: Time and Uncertainty:
Inference II

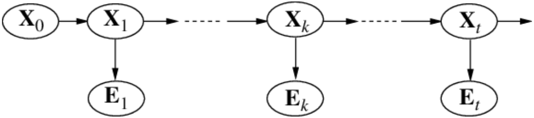
Where are we?

So far…

- Representing uncertainty in dynamic environments
- Reasoning in uncertain dynamic environments
  - Filtering and prediction
- Today: **Time and uncertainty: Inference II**

Smoothing

- Smoothing is computation of distribution of past states given current evidence, i.e. $\mathbf{P}(\mathbf{X}_k|\mathbf{e}_{1:t})$, $1 \leq k < t$



- Easiest to view as 2-step process (up to k, then $k+1$ to t)

$$\begin{aligned}
\mathbf{P}(\mathbf{X}_k|\mathbf{e}_{1:t}) &= \mathbf{P}(\mathbf{X}_k|\mathbf{e}_{1:k}, \mathbf{e}_{k+1:t}) && \text{(split notation)} \\
&= \alpha \mathbf{P}(\mathbf{X}_k|\mathbf{e}_{1:k})\mathbf{P}(\mathbf{e}_{k+1:t}|\mathbf{X}_k, \mathbf{e}_{1:k}) && \text{(Bayes)} \\
&= \alpha \mathbf{P}(\mathbf{X}_k|\mathbf{e}_{1:k})\mathbf{P}(\mathbf{e}_{k+1:t}|\mathbf{X}_k) && \text{(conditional independence)} \\
&= \alpha \mathbf{f}_{1:k}\mathbf{b}_{k+1:t}
\end{aligned}$$

- Here "backward" message is $\mathbf{b}_{k+1:t} = \mathbf{P}(\mathbf{e}_{k+1:t}|\mathbf{X}_k)$ analogous to forward message

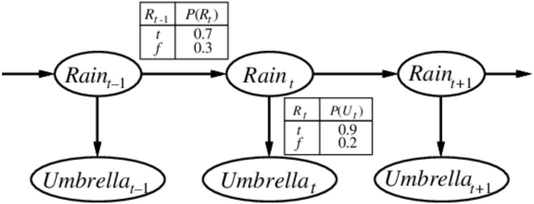
Smoothing

- Formula for backward message:

$$\mathbf{P}(\mathbf{e}_{k+1:t}|\mathbf{X}_k) = \sum_{\mathbf{x}_{k+1}} P(\mathbf{e}_{k+1}|\mathbf{x}_{k+1})P(\mathbf{e}_{k+2:t}|\mathbf{x}_{k+1})\mathbf{P}(\mathbf{x}_{k+1}|\mathbf{X}_k)$$

  (I'll show this is true shortly)
- First term is sensor model; third term is transition model; second is 'recursive call'
- Define $\mathbf{b}_{k+1:t} = \text{Backward}(\mathbf{b}_{k+2:t}, \mathbf{e}_{k+1:t})$
- The backward phase has to be initialised with $\mathbf{b}_{t+1:t} = \mathbf{P}(\mathbf{e}_{t+1:t}|\mathbf{X}_t) = \mathbf{1}$ (a vector of 1s) because probability of observing empty sequence is 1
- As before, all this is quite abstract, back to our example

Umbrella World: Compute $\mathbf{P}(R_1|u_1, u_2)$



We have $\mathbf{P}(R_1|u_1, u_2) = \alpha \mathbf{P}(R_1|u_1)\mathbf{P}(u_2|R_1)$
So we'll need to remind ourselves of $\mathbf{P}(R_1|u_1)$ from last lecture:

- $\mathbf{P}(R_1) = \sum_{r_0} \mathbf{P}(R_1|r_0)P(r_0) = \langle 0.7, 0.3 \rangle \times 0.5 + \langle 0.3, 0.7 \rangle \times 0.5 = \langle 0.5, 0.5 \rangle$
- Update with evidence $U_1 = true$ yields:

$$\mathbf{P}(R_1|u_1) = \alpha \mathbf{P}(u_1|R_1)\mathbf{P}(R_1) = \alpha \langle 0.9, 0.2 \rangle \langle 0.5, 0.5 \rangle \approx \langle 0.818, 0.182 \rangle$$

Smoothing Example Continued

$\mathbf{P}(R_1|u_1,u_2) = \alpha\mathbf{P}(R_1|u_1)\mathbf{P}(u_2|R_1)$

- Forward filtering process yielded $\langle 0.818, 0.182\rangle$ for first term
- The second term can be obtained through backward recursion:

$$\mathbf{P}(u_2|R_1) = \sum_{r_2} P(u_2|r_2)P(|r_2)\mathbf{P}(r_2|R_1)$$
$$= (0.9 \times 1 \times \langle 0.7, 0.3\rangle) + (0.2 \times 1 \times \langle 0.3, 0.7\rangle) = \langle 0.69, 0.41\rangle$$

- Plugged into the above equation this yields

$$\mathbf{P}(R_1|u_1,u_2) = \alpha\langle 0.818, 0.182\rangle \times \langle 0.69, 0.41\rangle \approx \langle 0.883, 0.117\rangle$$

- So our confidence that it rained on Day 1 increases when we see the umbrella on the second day as well as the first.
- A simple improved version of this that stores results runs in linear time (**forward-backward** algorithm)

Deriving the backward message

$$\mathbf{P}(\mathbf{e}_{k+1:t}|\mathbf{X}_k) = \sum_{\mathbf{x}_{k+1}} P(\mathbf{e}_{k+1}|\mathbf{x}_{k+1}) P(\mathbf{e}_{k+2:t}|\mathbf{x}_{k+1}) \mathbf{P}(\mathbf{x}_{k+1}|\mathbf{X}_k)$$

$$\mathbf{P}(\mathbf{e}_{k+1:t}|\mathbf{X}_k) = \sum_{\mathbf{x}_{k+1}} \mathbf{P}(\mathbf{e}_{k+1:t}, x_{k+1}|\mathbf{X}_k) \quad \text{(marginalisation)}$$

$$= \sum_{\mathbf{x}_{k+1}} \mathbf{P}(\mathbf{e}_{k+1}, \mathbf{e}_{k+2:t}, x_{k+1}|\mathbf{X}_k) \quad \text{(split notation)}$$

$$= \sum_{\mathbf{x}_{k+1}} P(\mathbf{e}_{k+1}|\mathbf{e}_{k+2:t}, x_{k+1}, \mathbf{X}_k) \mathbf{P}(\mathbf{e}_{k+2:t}, x_{k+1}|\mathbf{X}_k) \quad \text{(Bayes)}$$

$$= \sum_{\mathbf{x}_{k+1}} P(\mathbf{e}_{k+1}|x_{k+1}) \mathbf{P}(\mathbf{e}_{k+2:t}, x_{k+1}|\mathbf{X}_k) \quad \text{(independence)}$$

$$= \sum_{\mathbf{x}_{k+1}} P(\mathbf{e}_{k+1}|x_{k+1}) \mathbf{P}(\mathbf{e}_{k+2:t}|x_{k+1}, \mathbf{X}_k) \mathbf{P}(\mathbf{x}_{k+1}|\mathbf{X}_k) \quad \text{(Bayes)}$$

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\end{aligned}$$

Summary

- Hindsight computable via the forward backward algorithm.
- The equations involve recursion
  (as with filtering and prediction)
- Next time: **Time and Uncertainty: Inference III**
  - Finding the most likely sequence (Viterbi algorithm)