Informatics 2D: Reasoning and Agents

Alex Lascarides

School of informatics

The University of Edinburgh

Lecture 30b: Markov Decision Processes
Computing optimal policies

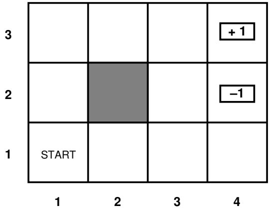
Where are we?

Last time. . .

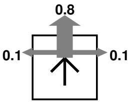
- Markov Decision Processes for representing sequential decision problems
- Optimal policy: mapping from state to action that maximises expected utility
- Today: **computing optimal policies**

Reminder of our example

- A 4x3-grid in which agent moves around (fully observable) and obtains utility of +1 or -1 in terminal states



(a)



(b)

- Actions are somewhat unreliable (in deterministic world, solution would be trivial)

Optimality in sequential decision problems

- MDPs very popular in various disciplines, different algorithms for finding optimal policies
- Before we present some of them, let us look at utility functions more closely
- We have used sum of rewards as utility of environment history until now, but what are the alternatives?
- First question: **finite horizon** or **infinite horizon**
- Finite means there is a fixed time N after which nothing matters:

$$\forall k\; U_h([s_0, s_1, \ldots, s_{N+k}]) = U_h([s_0, s_1, \ldots, s_N])$$

Optimality in sequential decision problems

- This leads to **non-stationary** optimal policies (N matters)
- With infinite horizon, we get **stationary** optimal policies (time at state doesn't matter)
- We are mainly going to use infinite horizon utility functions
- NOTE: sequences to terminal states can be finite even under infinite horizon utility calculation
- Second issue: how to calculate utility of sequences
- **Stationarity** here is reasonable assumption:

$$s_0 = s'_0 \wedge [s_0, s_1, s_2 \ldots] \succ [s'_0, s'_1, s'_2, \ldots] \Rightarrow [s_1, s_2 \ldots] \succ [s'_1, s'_2, \ldots]$$

Optimality in sequential decision problems

- Stationarity may look harmless, but there are only two ways to assign utilities to sequences under stationarity assumptions
- **Additive rewards**:

$$U_h([s_0, s_1, s_2 \ldots]) = R(s_0) + R(S_1) + R(S_2) + \ldots$$

- **Discounted rewards** (for **discount factor** $0 \leq \gamma \leq 1$)

$$U_h([s_0, s_1, s_2 \ldots]) = R(s_0) + \gamma R(S_1) + \gamma^2 R(S_2) + \ldots$$

- Discount factor makes more distant future rewards less significant
- We will mostly use discounted rewards in what follows

Optimality in sequential decision problems

- Choosing infinite horizon rewards creates a problem
- Some sequences will be infinite with infinite (additive) reward, how do we compare them?
- Solution 1: with discounted rewards the utility is bounded if single-state rewards are

$$U_h([s_0, s_1, s_2 \ldots]) = \sum_{t=0}^{\infty} \gamma^t R(s_t) \leq \sum_{t=0}^{\infty} \gamma^t R_{max} = R_{max}/(1-\gamma)$$

- Solution 2: under **proper policies**, i.e. if agent will eventually visit terminal state, additive rewards are finite
- Solution 3: compare **average reward** per time step

Value iteration

- **Value iteration** is an algorithm for calculating optimal policy in MDPs

  Calculate the utility of each state and then select optimal action based on these utilities

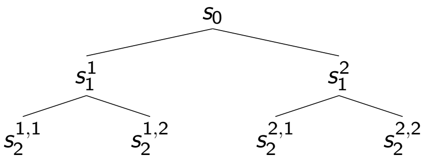
- Since discounted rewards seemed to create no problems, we will use

$$\pi^* = \arg\max_{\pi} E\left[\sum_{t=0}^{\infty} \gamma^t R(s_t)|\pi\right]$$

  as a criterion for optimal policy

Explaining $\pi^* = \arg\max_\pi E\left[\sum_{t=0}^{\infty} \gamma^t R(s_t) | \pi\right]$

- Each policy π yields a tree, with root node s_0, and daughters to a node s are the possible successor states given the action $\pi(s)$.
  - $T(s, a, s')$ gives the probability of traversing an arc from s to daughter s'.



- E is computed by:
  (a) For each path p in the tree, getting the product of the (joint) probability of the path in this tree with its discounted reward, and then
  (b) Summing over all the products from (a)
- So this is just a generalisation of single shot decision theory.

Utilities of states: : $U(s) \neq R(s)$!

- $R(s)$ is reward for being in s *now*.
- By making $U(s)$ the utility of the states that might follow it, $U(s)$ captures *long-term* advantages from being in s

  $U(s)$ *reflects what you can* do *from* s*;*
  $R(s)$ *does not.*

- States that follow depend on π. So utility of s given π is:

$$U^{\pi}(s) = E\left[\sum_{t=0}^{\infty} \gamma^t R(s_t) | \pi, s_0 = s\right]$$

- With this, "true" utility $U(s)$ is $U^{\pi^*}(s)$ (expected sum of discounted rewards if executing optimal policy)

Utilities in our example

- $U(s)$ computed for our example from algorithms to come.
- $\gamma = 1$, $R(s) = -0.04$ for nonterminals.

	1	2	3	4
3	0.812	0.868	0.918	+1
2	0.762		0.660	–1
1	0.705	0.655	0.611	0.388

Utilities of states

- Given $U(s)$, we can easily determine optimal policy:

$$\pi^*(s) = \arg\max_a \sum_{s'} T(s,a,s')U(s')$$

- Direct relationship between
utility of a state and that of its neighbours:
  Utility of a state is immediate reward plus expected utility of subsequent states if agent chooses optimal action

- This can be written as the famous **Bellman equations**:

$$U(s) = R(s) + \gamma \max_a \sum_{s'} T(s,a,s')U(s')$$

The value iteration algorithm

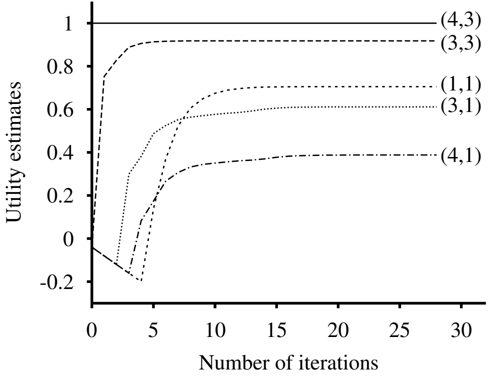
- For n states we have n Bellman equations with n unknowns (utilities of states)
- Value iteration is an iterative approach to solving the n equations.
- Start with arbitrary values and update them as follows:

$$U_{i+1}(s) \leftarrow R(s) + \gamma \max_a \sum_{s'} T(s,a,s') U_i(s')$$

- The algorithm converges to right and unique solution
- Like propagating values through network or utilities

The value iteration algorithm

- Value iteration in our example: evolution of utility values of states

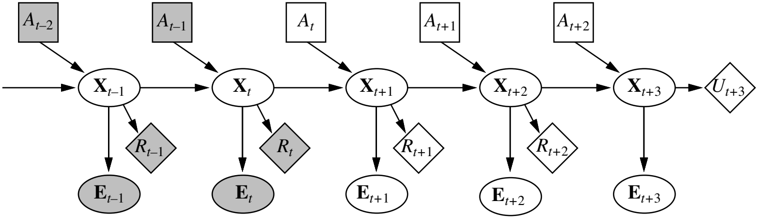


Decision-theoretic agents

- We now have (tediously) gathered all the ingredients to build decision-theoretic agents
- Transition and observation models will be described by a DBN
- They will be augmented by decision and utility nodes to obtain a **dynamic DN**
- Decisions will be made by projecting forward possible action sequences and choosing the best one
- Practical design for a **utility-based agent**

Decision-theoretic agents

- Dynamic decision networks look something like this
- General form of everything we have talked about in uncertainty part



Summary

- Sequential decision making
- Defined MDPs to model stochastic multi-step decision making processes
- Value iteration and policy iteration algorithms
- Design of decision-theoretic utility-based agents based on DDNs
- Completes our account of reasoning under uncertainty
- Next time: **AI and Ethics**