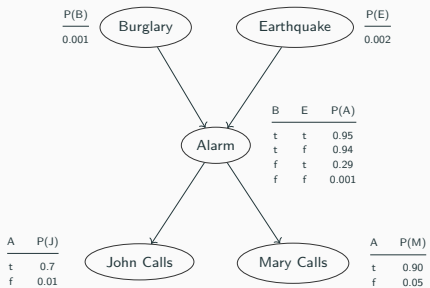


Lecture 24: Exact Inference in Bayesian Networks

Last Lecture Takeaway

Bayesian Networks allow a compact representation of Joint Probability Distributions.



$$P(B|j, m) = \langle 0.284, 0.716 \rangle$$

General Inference Procedure

- X - Query Variable
- E - Evidence Variables
- Y - Remaining Unobserved Values

$$P(X|e)$$

General Inference Procedure

- X - Query Variable
- E - Evidence Variables
- Y - Remaining Unobserved Values

$$P(X|e) = \alpha P(X, e)$$

General Inference Procedure

- X - Query Variable
- E - Evidence Variables
- Y - Remaining Unobserved Values

$$\begin{aligned} P(X|e) &= \alpha P(X, e) \\ &= \alpha \sum_y P(X, e, y) \end{aligned}$$

$$P(B|j, m)$$

$$= \alpha P(B, j, m)$$

$$= \alpha \sum_e \sum_a P(B, e, a, j, m)$$

$$= \alpha \sum_e \sum_a P(B)P(e)P(a|B, e)P(j|a)P(m|a)$$

$$= \alpha P(B) \sum_e P(e) \sum_a P(a|B, e)P(j|a)P(m|a)$$

$$P(B|j, m)$$

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$$P(B|j, m)$$

$$= \alpha P(B, j, m)$$

$$= \alpha \sum_e \sum_a P(B, e, a, j, m)$$

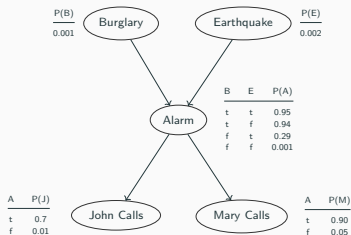
$$= \alpha \sum_e \sum_a P(B)P(e)P(a|B, e)P(j|a)P(m|a)$$

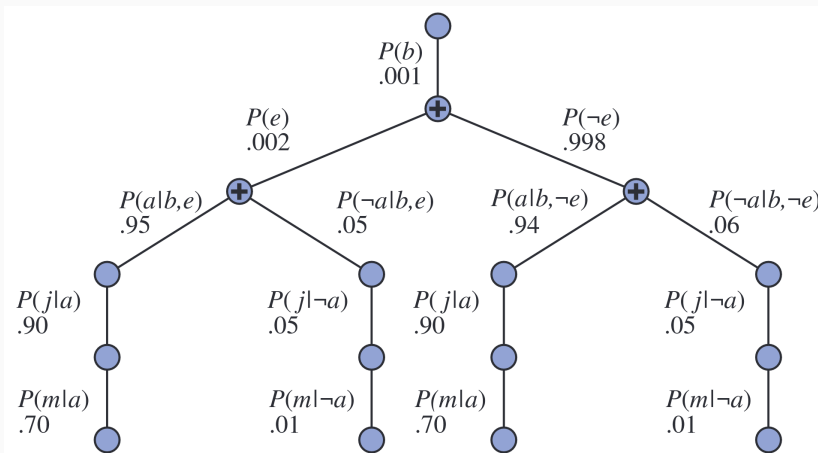
$$= \alpha P(B) \sum_e P(e) \sum_a P(a|B, e)P(j|a)P(m|a)$$

$$\begin{aligned}
& P(B|j, m) \\
&= \alpha P(B, j, m) \\
&= \alpha \sum_e \sum_a P(B, e, a, j, m) \\
&= \alpha \sum_e \sum_a P(B)P(e)P(a|B, e)P(j|a)P(m|a) \\
&= \alpha P(B) \sum_e P(e) \sum_a P(a|B, e)P(j|a)P(m|a)
\end{aligned}$$

$$\begin{aligned}
& P(B|j, m) \\
&= \alpha P(B, j, m) \\
&= \alpha \sum_e \sum_a P(B, e, a, j, m) \\
&= \alpha \sum_e \sum_a P(B)P(e)P(a|B, e)P(j|a)P(m|a) \\
&= \alpha P(B) \sum_e P(e) \sum_a P(a|B, e)P(j|a)P(m|a)
\end{aligned}$$

$$\begin{aligned}
& P(b) \sum_e P(e) \sum_a P(a|b, e) P(j|a) P(m|a) \\
&= 0.001 * (0.002 * (0.95 * 0.9 * 0.7 + 0.05 * 0.05 * 0.01) \\
&+ 0.998 * (0.94 * 0.90 * 0.7 + 0.06 * 0.05 * 0.01)) \\
&= 0.00059224259
\end{aligned}$$





Variable Elimination Algorithm

$$P(B|j, m) = \alpha P(B) \sum_e P(e) \sum_a P(a|B, e) P(j|a) P(m|a)$$

Variable Elimination Algorithm

$$P(B|j, m) = \alpha \underbrace{P(B)}_{f_1(B)} \sum_e \underbrace{P(e)}_{f_2(E)} \sum_a \underbrace{P(a|B, e)}_{f_3(A, B, E)} \underbrace{P(j|a)}_{f_4(A)} \underbrace{P(m|a)}_{f_5(A)}$$

Variable Elimination Algorithm

$$P(B|j, m) = \alpha \underbrace{P(B)}_{f_1(B)} \sum_e \underbrace{P(e)}_{f_2(E)} \sum_a \underbrace{P(a|B, e)}_{f_3(A, B, E)} \underbrace{P(j|a)}_{f_4(A)} \underbrace{P(m|a)}_{f_5(A)}$$

$$f_5(A) = \langle P(m|a), P(m|\neg a) \rangle$$

Variable Elimination Algorithm

$$P(B|j, m) = \alpha f_1(B) \times \sum_e f_2(E) \sum_a f_3(A, B, E) \times f_4(A) \times f_5(A)$$

Variable Elimination Algorithm

$$P(B|j, m) = \alpha f_1(B) \times \sum_e f_2(E) \sum_a f_3(A, B, E) \times f_4(A) \times f_5(A)$$

Variable Elimination Algorithm

$$P(B|j, m) = \alpha f_1(B) \times \sum_e f_2(E) \sum_a f_3(A, B, E) \times f_4(A) \times f_5(A)$$

$$f_6(B, E) = \sum_a f_3(A, B, E) \times f_4(A) \times f_5(A)$$

Variable Elimination Algorithm

$$P(B|j, m) = \alpha f_1(B) \times \sum_e f_2(E) \sum_a f_3(A, B, E) \times f_4(A) \times f_5(A)$$

$$\begin{aligned} f_6(B, E) &= \sum_a f_3(A, B, E) \times f_4(A) \times f_5(A) \\ &= (f_3(a, B, E) \times f_4(a) \times f_5(a)) + \\ &\quad (f_3(\neg a, B, E) \times f_4(\neg a) \times f_5(\neg a)) \end{aligned}$$

Variable Elimination Algorithm

$$P(B|j, m) = \alpha f_1(B) \times \sum_e f_2(E) \times f_6(B, E)$$

Variable Elimination Algorithm

$$P(B|j, m) = \alpha f_1(B) \times \sum_e f_2(E) \times f_6(B, E)$$

Variable Elimination Algorithm

$$P(B|j, m) = \alpha f_1(B) \times \sum_e f_2(E) \times f_6(B, E)$$

$$f_7(B) = \sum_e f_2(E) \times f_6(B, E)$$

Variable Elimination Algorithm

$$P(B|j, m) = \alpha f_1(B) \times \sum_e f_2(E) \times f_6(B, E)$$

$$\begin{aligned} f_7(B) &= \sum_e f_2(E) \times f_6(B, E) \\ &= (f_2(e) \times f_6(B, e)) + \\ &\quad (f_2(\neg e) \times f_6(B, \neg e)) \end{aligned}$$

$$P(B|j, m) = \alpha f_1(B) \times f_7(B)$$

Variable Elimination Algorithm

Pointwise Product

$$f_1(A, B) \times f_2(B, C) = f(A, B, C)$$

A	B	$f_1(A, B)$	B	C	$f_2(B, C)$
T	T	0.3	T	T	0.2
T	F	0.7	T	F	0.8
F	T	0.9	T	T	0.6
F	F	0.1	F	F	0.4

A	B	C	$f(A, B, C)$
T	T	T	0.3×0.2
T	T	F	0.3×0.8
T	F	T	0.7×0.6
T	F	F	0.7×0.4
F	T	T	0.9×0.2
F	T	F	0.9×0.8
F	F	T	0.1×0.6
F	F	F	0.1×0.4

Summing Out

$$\sum_e f_2(E) \times f_3(A, B, E) \times f_4(A) \times f_5(A)$$

Summing Out

$$\begin{aligned} & \sum_e f_2(E) \times f_3(A, B, E) \times f_4(A) \times f_5(A) \\ &= f_4(A) \times f_5(A) \times \sum_e f_2(E) \times f_3(A, B, E) \end{aligned}$$

Variable Elimination Example

$$\begin{aligned}P(J|b) &= \alpha \sum_e \sum_a \sum_m P(J, b, e, a, m) \\&= \alpha \sum_e \sum_a \sum_m P(b)P(e)P(a|b, e)P(J|a)P(m|a) \\&= \alpha' \sum_e P(e) \sum_a P(a|b, e)P(J|a) \sum_m P(m|a) \\&= \alpha' \sum_e f_1(E) \sum_a f_2(A, E)f_3(J, A) \\&= \alpha' \sum_e f_1(E)f_4(J, E) \\&= \alpha' f_5(J)\end{aligned}$$

Variable Elimination Example

$$\begin{aligned}P(J|b) &= \alpha \sum_e \sum_a \sum_m P(J, b, e, a, m) \\&= \alpha \sum_e \sum_a \sum_m P(b)P(e)P(a|b, e)P(J|a)P(m|a) \\&= \alpha' \sum_e P(e) \sum_a P(a|b, e)P(J|a) \sum_m P(m|a) \\&= \alpha' \sum_e f_1(E) \sum_a f_2(A, E)f_3(J, A) \\&= \alpha' \sum_e f_1(E)f_4(J, E) \\&= \alpha' f_5(J)\end{aligned}$$

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Variable Elimination Example

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Variable Elimination Example

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Variable Elimination Example

$$\begin{aligned}P(J|b) &= \alpha \sum_e \sum_a \sum_m P(J, b, e, a, m) \\&= \alpha \sum_e \sum_a \sum_m P(b)P(e)P(a|b, e)P(J|a)P(m|a) \\&= \alpha' \sum_e \underbrace{P(e)}_{f_1(E)} \sum_a \underbrace{P(a|b, e)}_{f_2(A, E)} \underbrace{P(J|a)}_{f_3(J, A)} \underbrace{\sum_m P(m|a)}_{=1} \\&= \alpha' \sum_e f_1(E) \sum_a f_2(A, E) f_3(J, A) \\&= \alpha' \sum_e f_1(E) f_4(J, E) \\&= \alpha' f_5(J)\end{aligned}$$

$$P(B|j, m) = \alpha f_1(B) \times \sum_e f_2(E) \sum_a f_3(A, B, E) \times f_4(A) \times f_5(A)$$

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- Exact Inference in Bayesian Networks
- Inference by Enumeration
- Variable Elimination algorithm