Lecture 26: Time and Uncertainty (Part 1)

Last Lecture

Exact & Approximate Inference for **Static** *Environments*

Reminder: Probabilistic Toolbox

Product/Bayes Rule:

$$P(A, B) = P(A|B)P(B) = P(B|A)P(A)$$

Marginalization:

$$P(Y) = \sum_z P(Y, z)$$

Normalization:

$$P(X|e) = \alpha P(X, e)$$

Uncertain Processes over Time

E_t — Observed variables at time t

X_t — Non-observable variables at time t

States start at $t = 0$.

Evidence starts arriving $t = 1$

Example

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$$E = \{U_1, U_2, U_3, U_4, \dots\}$$

$$U_{1:3} = U_1, U_2, U_3$$

Assumption 1: Stationary Processes

Stationary Process: The laws governing change don't change over time.

$P(U_t | Parents(U_t))$ does not depend on t

Assumption 2: Markov Assumption

Markov Assumption: The current state only depends on a finite history of previous states.

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First-order Markov process:

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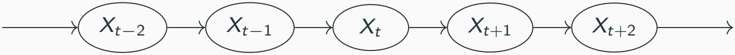
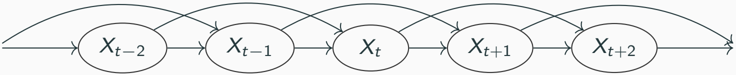


Assumption 2: Markov Assumption

Second-order Markov process:

$$P(X_t|X_{0:t-1}) = P(X_t|X_{t-1}, X_{t-2})$$



Assumption 3: Sensor Model

Evidence variables are conditionally independent of all other variables given the current state:

$$P(E_t|X_{0:t}, E_{0:t-1}) = P(E_t|X_t)$$

Transition Model: $P(X_t|X_{t-1})$

Sensor Model: $P(E_t|X_t)$

Prior: $P(X_0)$

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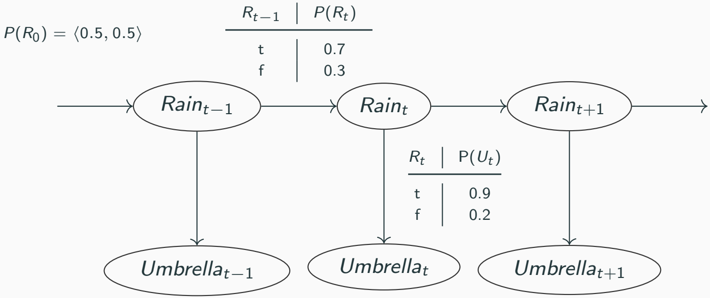
$$P(X_0, X_1, \ldots, X_t, E_1, \ldots, E_t) = \underbrace{P(X_0)}_{\text{prior}} \prod_{i=1}^{t} \underbrace{P(X_i|X_{i-1})}_{\text{transition}} \underbrace{P(E_i|X_i)}_{\text{sensor}}$$

Umbrella World



Inference Tasks in Temporal Models

- **Filtering/Monitoring**: $P(X_t|e_{1:t})$
- **Prediction**: $P(X_{t+k}|e_{1:t})$
- **Evidence Likelihood**: $P(e_{1:t})$
- **Smoothing/Hindsight**: $P(X_k|e_{1:t}), \quad 0 \leq k < t$
- **Most likely explanation:** $\arg\max_{x_{1:t}} P(x_{1:t}|e_{1:t})$

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Filtering

Recursive Estimation:

$$P(X_{t+1}|e_{1:t+1}) = f(e_{t+1}, P(X_t|e_{1:t}))$$

$$
\begin{aligned}
P(X_{t+1}|e_{1:t+1}) &= P(X_{t+1}|e_{1:t}, e_{t+1}) && \text{(split notation)} \\
&= \alpha P(X_{t+1}, e_{1:t}, e_{t+1}) && \text{(Bayes)} \\
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Alternative View: Message Passing

$$f_{1:t} = P(X_t | e_{1:t})$$

$$f_{1:t+1} = P(X_{t+1} | e_{1:t+1}) = \alpha \text{Forward}(f_{1:t}, e_{t+1})$$

Example

$$\mathbf{P(R_2|U_1 = true, U_2 = true)}$$

$$
\begin{aligned}
P(R_0) &= \langle 0.5, 0.5\rangle \\
P(R_2|u_1, u_2) &= \alpha P(u_2|R_2) \sum_{r_1} P(R_2|r_1) P(r_1|u_1) \\
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&= \alpha' \langle 0.9, 0.2\rangle (\langle 0.7, 0.3\rangle \times 0.5 \\
&\quad + \langle 0.3, 0.7\rangle \times 0.5) \\
&= \alpha' \langle 0.9, 0.2\rangle \langle 0.5, 0.5\rangle \\
&= \langle 0.818, 0.182\rangle \\
P(R_2|u_1, u_2) &= \alpha \langle 0.9, 0.2\rangle (\langle 0.7, 0.3\rangle \times 0.818 \\
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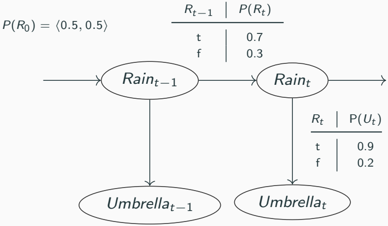
&= \alpha \langle 0.9, 0.2\rangle \langle 0.627, 0.373\rangle \\
&= \alpha \langle 0.565, 0.075\rangle \\
&= \langle 0.883, 0.117\rangle
\end{aligned}
$$



Example

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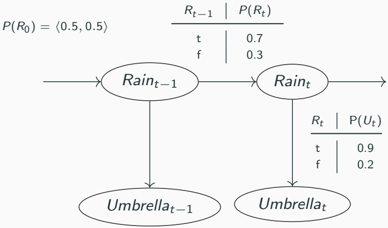

Example

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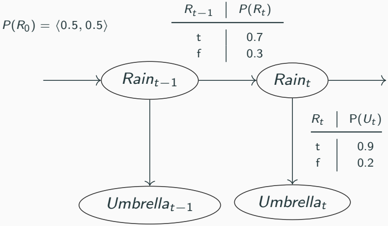

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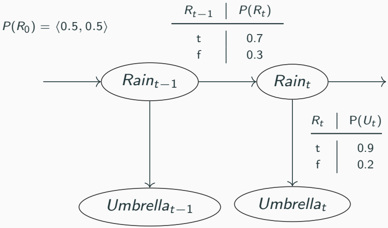

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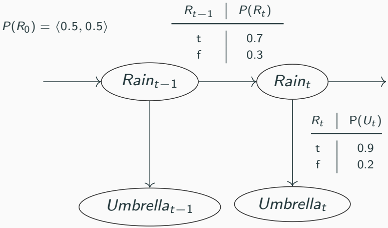

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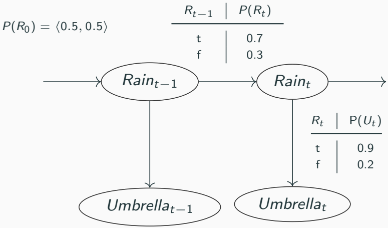

Prediction

$$P(X_{t+k+1}|e_{1:t})$$

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$$
\begin{aligned}
&P(X_{t+k+1}|e_{1:t}) \\
&= \sum_{x_{t+k}} P(X_{t+k+1}, x_{t+k}|e_{1:t}) \\
&= \sum_{x_{t+k}} P(X_{t+k+1}|x_{t+k}, e_{1:t})P(x_{t+k}|e_{1:t}) \\
&\sum_{x_{t+k}} P(X_{t+k+1}|x_{t+k})P(x_{t+k}|e_{1:t})
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$$

(Extra) Likelihood of Evidence

$$\mathbf{P}(\mathbf{e_{1:t}})$$

Exercise: Can we derive a recursive computation for this too?

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Exercise: Can we derive a recursive computation for this too?

$$P(e_{1:t}) = \sum_{x_t} P(x_t, e_{1:t})$$

$$l_{1:t} = P(X_t, e_{1:t})$$

$$l_{1:t+1} = Forward(l_{1:t}, e_{t+1})$$

$$L_{1:t} = P(e_{1:t}) = \sum_{x_t} l_{1:t}(x_t, e_{1:t})$$

Summary

- Time and Uncertainty
- Stationarity and Markov Assumptions
- Filtering, Prediction, Likelihood