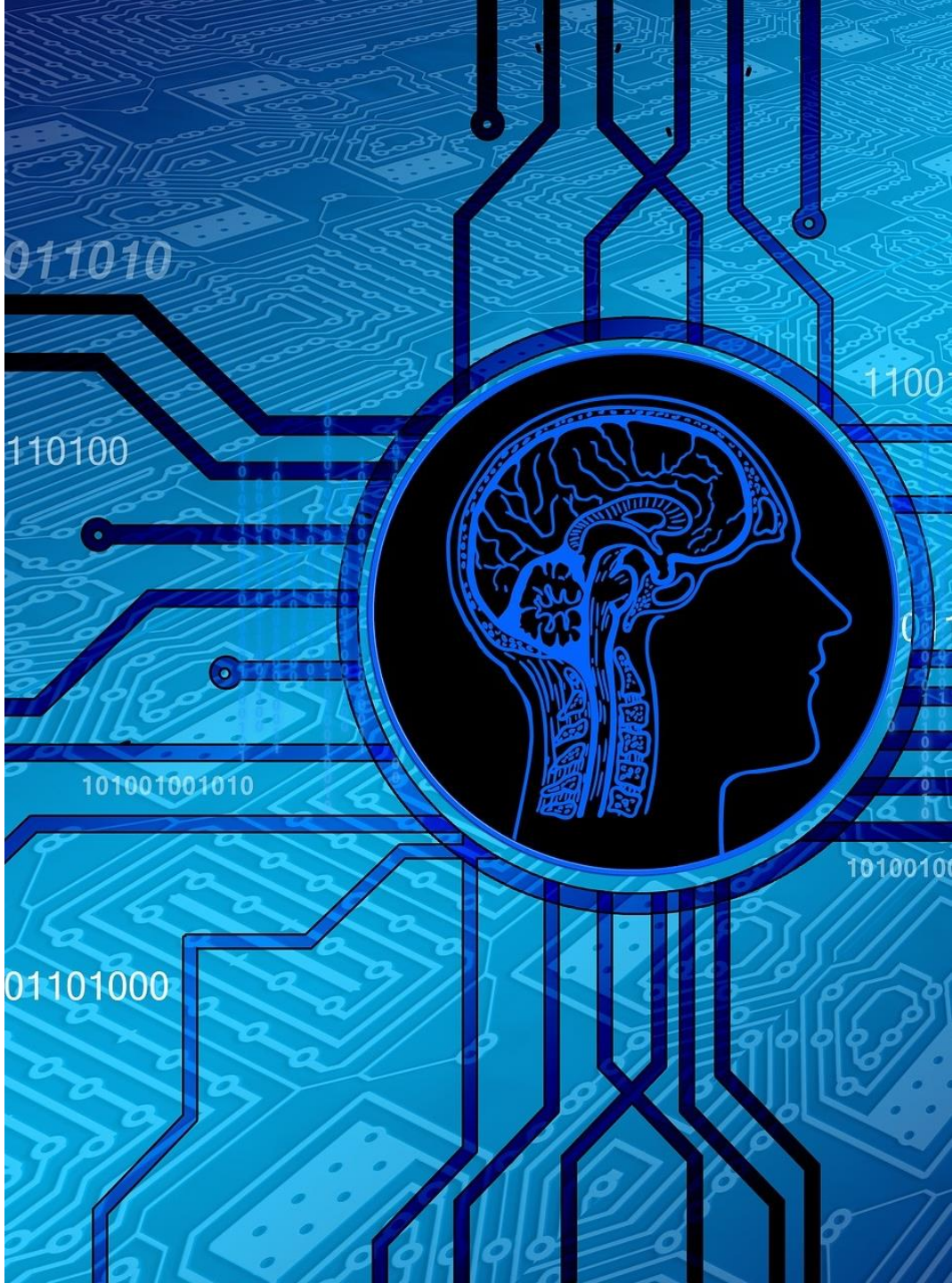
Unification & Generalised Modus Ponens

Informatics 2D: Reasoning and Agents



Propositional vs First-Order Inference

- So far, we know how to formulate simple inference rules in FOL.
- Goal: Enabling first-order inference
- Idea:
  - Convert the KB to propositional logic and use propositional inference
- Better idea:
  - Use inference methods to work with first-order sentences directly

Inference Rules for Quantifiers

Universal instantiation (UI)

- Infer any sentence by substituting a ground term for the variable

$$\frac{\forall v \;\; \alpha}{\text{SUBST}(\{v/g\}, \alpha)}$$

Example: $\forall x.\ King(x) \wedge Greedy(x) \Rightarrow Evil(x)$ yields:

- $King(John) \wedge Greedy(John) \Rightarrow Evil(John)$
- $King(Richard) \wedge Greedy(Richard) \Rightarrow Evil(Richard)$
- $King(Father(John)) \wedge Greedy(Father(John)) \Rightarrow Evil(Father(John))$

Inference Rules for Quantifiers

Existential instantiation (EI)

- Replace the variable by a single new constant symbol

$$\frac{\exists v \;\; \alpha}{\text{SUBST}(\{v/k\}, \alpha)}$$

Example. $\exists x.\ Crown(x) \wedge OnHead(x, John)$ yields:

- $Crown(C_1) \wedge OnHead(C_1, John)$

provided C_1 is a new constant symbol, called a Skolem constant

Inferential Equivalence

- UI can be applied many times to produce many different outcomes
- EI can be applied once, then the existentially quantified sentence could be discarded.
- The new knowledge base (KB') is inferentially equivalent to the old KB

Reduction to propositional inference

- Suppose the KB contains just the following:

$\forall x.\ King(x) \wedge Greedy(x) \Rightarrow Evil(x)$ $King(John)$ $Greedy(John)$ $Brother(Richard, John)$

- Instantiating the universal sentence in all possible ways, we have:
  - $King(John) \wedge Greedy(John) \Rightarrow Evil(John)$
  - $King(Richard) \wedge Greedy(Richard) \Rightarrow Evil(Richard)$
  - $King(John)$
  - $Greedy(John)$
  - $Brother(Richard, John)$

Reduction to propositional inference

- Suppose the KB contains just the following:

$\forall x.\ King(x) \wedge Greedy(x) \Rightarrow Evil(x)$ $\quad King(John)$ $\quad Greedy(John)$ $\quad Brother(Richard, John)$

- Instantiating the universal sentence in all possible ways, we have:
  - $King(John) \wedge Greedy(John) \Rightarrow Evil(John)$
  - $King(Richard) \wedge Greedy(Richard) \Rightarrow Evil(Richard)$
  - $King(John)$
  - $Greedy(John)$
  - $Brother(Richard, John)$

KB': The new KB will then include extra propositional symbols

Propositionalization

- Every FOL KB can be propositionalized so as to preserve entailment
  - A ground sentence is entailed by new KB iff entailed by original KB

- Idea: propositionalize KB and query, apply DPLL (or some other complete propositional method), return result

- Problem: with function symbols, there are infinitely many ground terms,
  - e.g., *Father*(*Father*(*Father*(*John*)))

Theorem: Herbrand (1930)

- If a sentence α is entailed by a FOL KB, it is entailed by a finite subset of the propositionalized KB

Idea: For $n = 0$ to ∞ do
- create a propositional KB by instantiating with depth-n terms
- see if α is entailed by this KB

Problem: works if α is entailed, loops forever if α is not entailed

Theorem: Turing (1936), Church (1936).

- Entailment for FOL is **semi-decidable**
  (i.e., algorithms exist that say yes to every entailed sentence, but no algorithm exists that also says no to every non-entailed sentence.)

A first-order inference rule

Problems with Propositionalization

$\forall x.\ \text{King}(x) \wedge \text{Greedy}(x) \Rightarrow \text{Evil}(x)$ $\quad$ $\text{King}(\text{John})$

$\forall y.\ \text{Greedy}(y)$ $\quad$ $\text{Brother}(\text{Richard},\text{John})$

- It seems obvious that *Evil*(*John*), but propositionalization produces lots of facts such as *Greedy*(*Richard*) that are irrelevant.

- With p k-ary predicates and n constants, there are $p \cdot n^k$ instantiations.
- We want to find a substitution both for the variables in the implication sentence and for the variables in the sentences in the KB (e.g., *x/John, y/John*).

Modus Ponens (Propositional Logic)

Latin for "*method of putting by placing*" – "*way that affirms by affirming*"

$$\frac{P \qquad P \Longrightarrow Q}{Q}$$

$$P, \qquad P \Longrightarrow Q \vdash Q$$

Generalized Modus Ponens (GMP)

such that $\text{SUBST}(\theta, p_i') = \text{SUBST}(\theta, p_i)$, for all i,

$$\frac{p_1',\ p_2',\ \ldots,\ p_n',\ (p_1 \wedge p_2 \wedge \ldots \wedge p_n \Rightarrow q)}{\text{SUBST}(\theta, q)}$$

KB

$\forall x\ \ King(x) \wedge Greedy(x) \Rightarrow Evil(x)$
$King(John)$
$\forall y\ \ Greedy(y)$

Applying GMP to KB

p_1' is $King(John)$ — p_1 is $King(x)$
p_2' is $Greedy(y)$ — p_2 is $Greedy(x)$
θ is $\{x/John, y/John\}$ — q is $Evil(x)$
$\text{SUBST}(\theta, q)$ is $Evil(John)$

- GMP is a sound inference rule.

Unification

MAKE DIFFERENT LOGICAL EXPRESSIONS LOOK IDENTICAL

Unification

- The `UNIFY` algorithm takes two sentences and returns a unifier for them if one exists.

$$\textsc{Unify}(p, q) = \theta \text{ where } \textsc{Subst}(\theta, p) = \textsc{Subst}(\theta, q)$$

Unification examples

α	β	θ
Knows(John, x)	*Knows(John, Jane)*	
Knows(John, x)	*Knows(y, OJ)*	
Knows(John, x)	*Knows(y, Mother(y))*	
Knows(John, x)	*Knows(x, Richard)*	

Unification examples

α	β	θ
Knows(John, x)	*Knows(John, Jane)*	{*x/Jane*}
Knows(John, x)	*Knows(y, OJ)*	
Knows(John, x)	*Knows(y, Mother(y))*	
Knows(John, x)	*Knows(x, Richard)*	

Unification examples

α	β	θ
Knows(John, x)	*Knows(John, Jane)*	{*x/Jane*}
Knows(John, x)	*Knows(y, OJ)*	{*x/OJ, y/John*}
Knows(John, x)	*Knows(y, Mother(y))*	
Knows(John, x)	*Knows(x, Richard)*	

Unification examples

α	β	θ
Knows(John, x)	*Knows(John, Jane)*	{*x/Jane*}
Knows(John, x)	*Knows(y, OJ)*	{*x/OJ, y/John*}
Knows(John, x)	*Knows(y, Mother(y))*	{*y/John, x/Mother(John)*}
Knows(John, x)	*Knows(x, Richard)*	

Unification examples

α	β	θ
Knows(John, x)	*Knows(John, Jane)*	{*x/Jane*}
Knows(John, x)	*Knows(y, OJ)*	{*x/OJ, y/John*}
Knows(John, x)	*Knows(y, Mother(y))*	{*y/John, x/Mother(John)*}
Knows(John, x)	*Knows(x, Richard)*	**Fail!**

Unification examples

α	β	θ
Knows(John, x)	*Knows(John, Jane)*	{*x/Jane*}
Knows(John, x)	*Knows(y, OJ)*	{*x/OJ, y/John*}
Knows(John, x)	*Knows(y, Mother(y))*	{*y/John, x/Mother(John)*}
Knows(John, x)	*Knows(x, Richard)*	**Fail!**

Standardizing variables apart eliminates overlap of variables

e.g. change *Knows(x, Richard)* to *Knows(z_{17}, Richard)* and then we succeed the last case with $\theta = \{z_{17}/John, x/Richard\}$

Most General Unifier (MGU)

Unifying *Knows(John, x)* and *Knows(y, z)*

$$\theta = \{y/John, x/z\} \quad \text{or} \quad \theta = \{y/John, x/John, z/John\}$$

The first unifier is more general than the second.

FOL: There is a **single** most general unifier (MGU) that is unique up to renaming of variables.

$$\text{MGU} = \{y/John, x/z\}$$

Can be viewed as an equation solving problem.

- *i.e. solve Knows(John, x)* $\stackrel{?}{=}$ *Knows(y, z)*

MGU Examples

	MGU
Loves(John, x) $\stackrel{?}{=}$ *Loves(y, Mother(y))*	
Loves(John, Mother(y)) $\stackrel{?}{=}$ *Loves(y, y)*	

MGU Examples

	MGU
$Loves(John, x) \stackrel{?}{=} Loves(y, Mother(y))$	$\{x/Mother(John), y/John\}$
$Loves(John, Mother(y)) \stackrel{?}{=} Loves(y, y)$	

MGU Examples

	MGU
$Loves(John, x) \stackrel{?}{=} Loves(y, Mother(y))$	$\{x/Mother(John), y/John\}$
$Loves(John, Mother(y)) \stackrel{?}{=} Loves(y, y)$	Fail!

Finding the MGU

Can be broken-down into a series of steps

- Decomposition
- Conflict
- Eliminate
- Delete
- Switch
- Coalesce
- Occurs Check

Other presentations of algorithm are possible (see R&N)

Decomposition

Given

$f(s_1, \ldots, s_n) \stackrel{?}{=} f(t_1, \ldots, t_n)$

Replace with

$s_1 \stackrel{?}{=} t_1, \ldots, s_n \stackrel{?}{=} t_n$

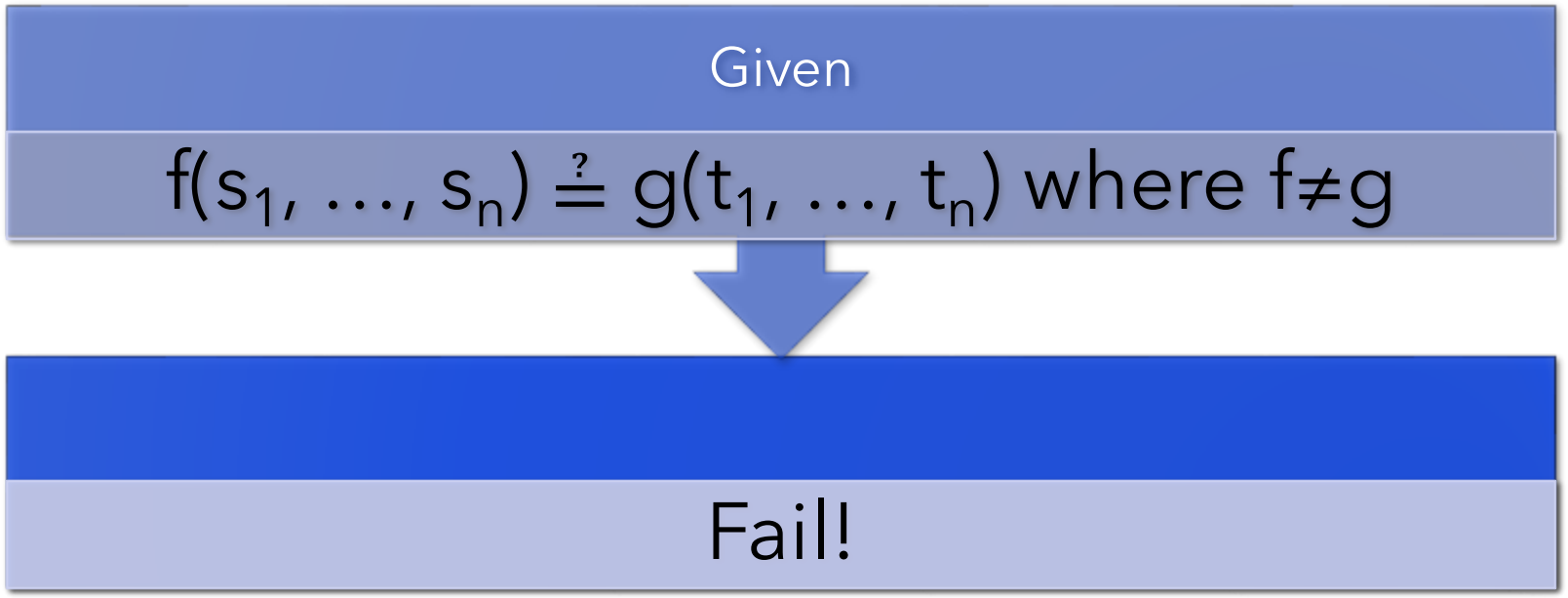
Example

Given

$Knows(John, x) \stackrel{?}{=} Knows(y, z)$

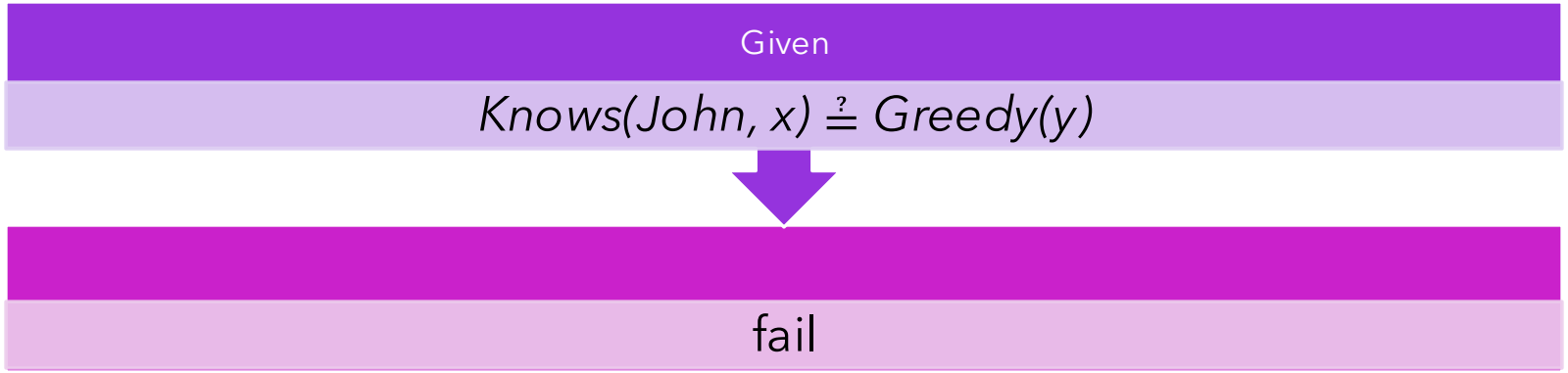
Replace with

$John \stackrel{?}{=} y, \quad x \stackrel{?}{=} z$



Conflict

Example



Eliminate

Given

P, $x \stackrel{?}{=} t$ where x occurs in P but not in t, and t is not a variable

Replace with

$P\{x/t\}$ and $x \stackrel{?}{=} t$

Example

Given

$Knows(John, x) \stackrel{?}{=} Knows(y, z)$, $z \stackrel{?}{=} Richard$

Replace with

$Knows(John, x) \stackrel{?}{=} Knows(y, Richard)$, $z \stackrel{?}{=} Richard$

Delete

Given

$P,\ s \stackrel{?}{=} s$

Replace with

P

Example

Given

$z \stackrel{?}{=} Richard,\quad Greedy(John) \stackrel{?}{=} Greedy(John)$

Replace with

$z \stackrel{?}{=} Richard$

Given

$P,\ s \stackrel{?}{=} x$ where x is a variable and s is not

Replace with

P and $x \stackrel{?}{=} s$

Switch

Example

Given

$Knows(John, x) \stackrel{?}{=} Knows(y, z),\quad Richard \stackrel{?}{=} z$

Replace with

$Knows(John, x) \stackrel{?}{=} Knows(y, z),\quad z \stackrel{?}{=} Richard$

Coalesce

Given

$P,\ x \stackrel{?}{=} y$ where x, y variables occurring in P

Replace with

$P\{x/y\}$ and $x \stackrel{?}{=} y$

Example

Given

$Knows(John, x) \stackrel{?}{=} Knows(y, z),\quad y \stackrel{?}{=} z$

Replace with

$Knows(John, x) \stackrel{?}{=} Knows(z, z),\quad y \stackrel{?}{=} z$

Occurs Check

Given

$x \stackrel{?}{=} s$ where x **occurs** in s and s not a variable

Fail!

Example

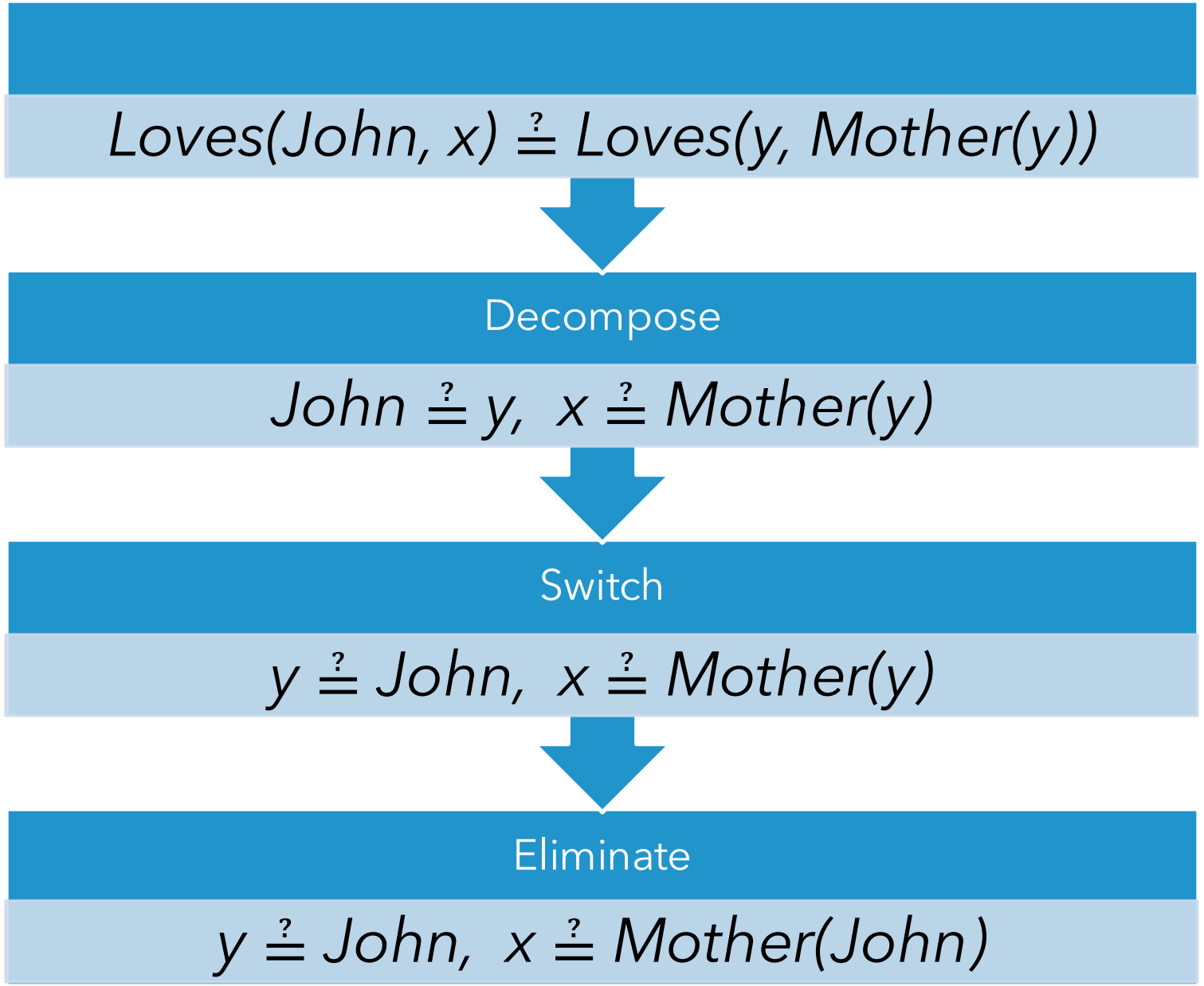
Given

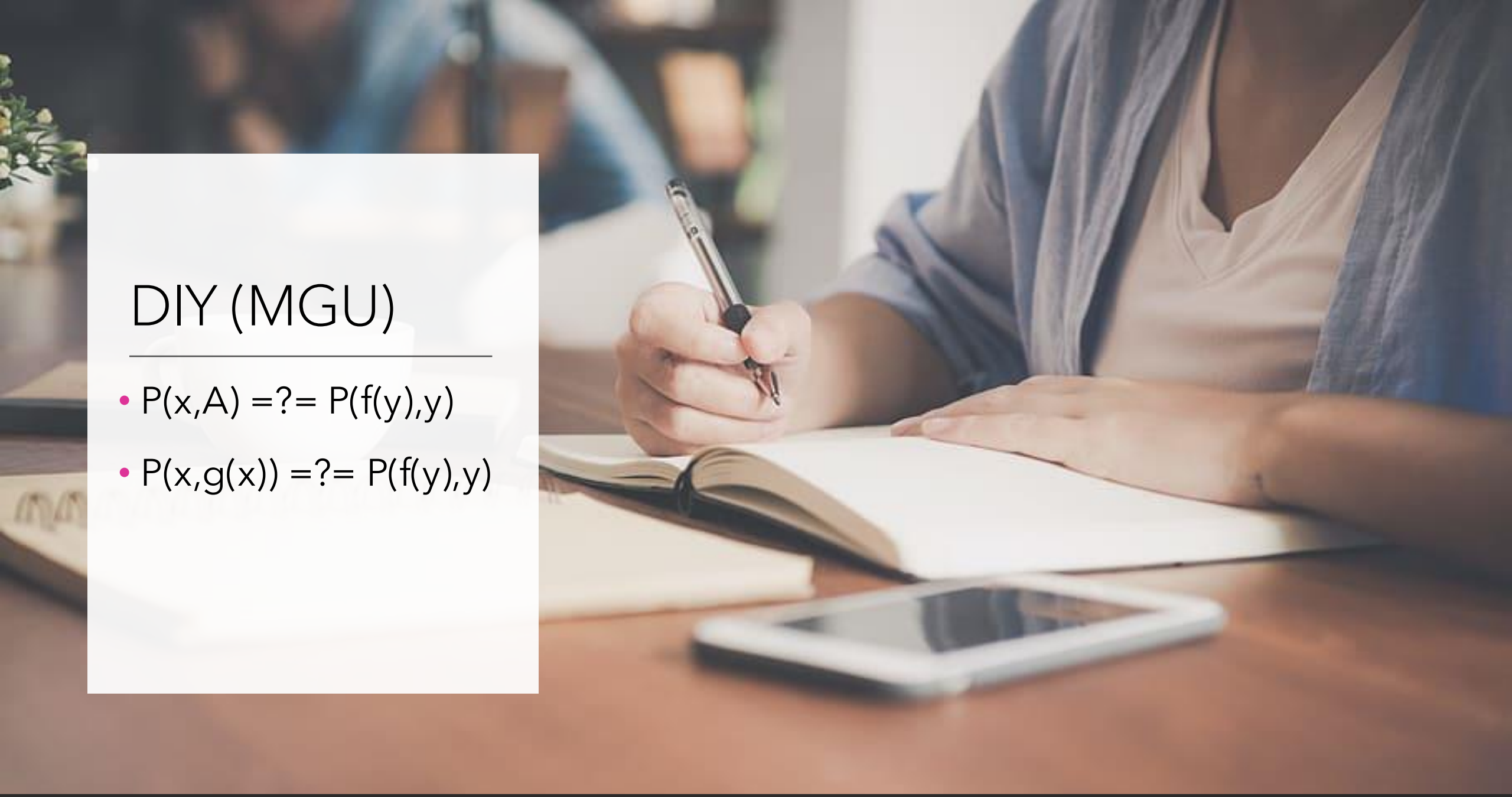
$P(x), \quad x \stackrel{?}{=} Father(x)$

Fail (else Eliminate will loop)

$P(Father(Father(Father(\ldots))))$

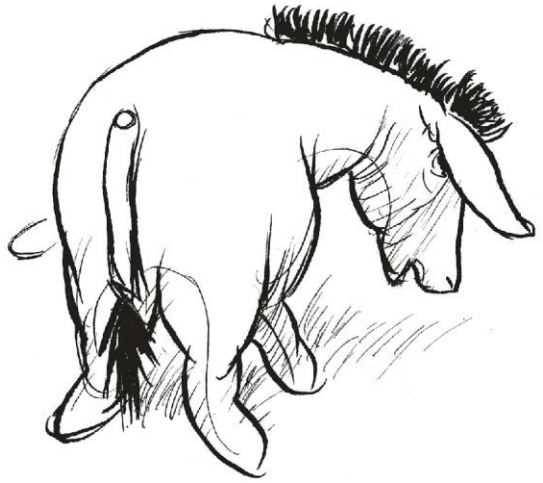
Example


Loves(John, x) ≟ Loves(y, Mother(y))
Decompose
John ≟ y, x ≟ Mother(y)
Switch
y ≟ John, x ≟ Mother(y)
Eliminate
y ≟ John, x ≟ Mother(John)



DIY (MGU)

- P(x,A) =?= P(f(y),y)
- P(x,g(x)) =?= P(f(y),y)



New Example KB

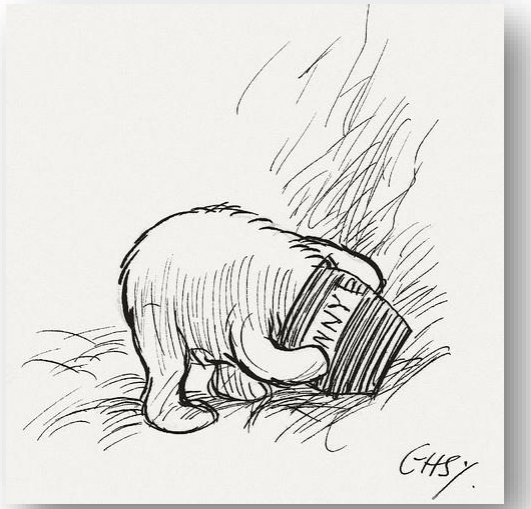
Example Knowledge Base

It is known in The Hundred-Acre Wood that if someone who is very fond of food gives a treat to one of their friends, they are really generous.

Eeyore, the sad donkey, has some hunny that he has received for his birthday from Winnie-the-Pooh, who, as we know, is very fond of food.

Prove that Winnie-the-Pooh is generous.

Formalisation



if someone who is very fond of food gives a treat to one of their friends, they are really generous

- $VeryFondOfFood(x) \wedge Treat(y) \wedge Friend(z) \wedge Gives(x, y, z) \Rightarrow Generous(x)$

Eeyore (…) has some hunny

- $\exists x. Owns(Eeyore, x) \wedge Hunny(x)$ or after EI: $Owns(Eeyore, H_1) \wedge Hunny(H_1)$

that he has received for his birthday from Winnie-the-Pooh

- $Hunny(x) \wedge Owns(Eeyore, x) \Rightarrow Gives(Pooh, x, Eeyore)$

Hunny is a treat.

- $Hunny(x) \Rightarrow Treat(x)$

Residents of the the Hundred-Acre Wood are friends.

- $Resident(x, HundredAcreWood) \Rightarrow Friend(x)$

Eeyore is a resident of the the Hundred-Acre Wood.

- $Resident(Eeyore, HundredAcreWood)$

Pooh is very fond of food.

- $VeryFondOfFood(Pooh)$

Why?

- Setting the scene for inference & resolution.
- Linked to logic programming.

…but more in the next lecture!