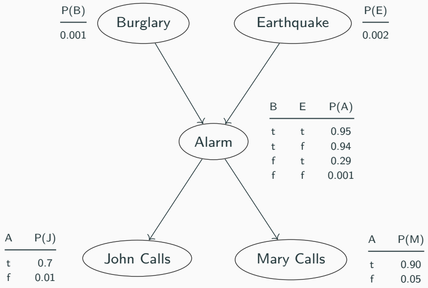
Lecture 23: Exact Inference in Bayesian Networks

Last Lecture Takeaway

Bayesian Networks allow a compact representation of Joint Probability Distributions.



$$P(B|j, m)$$
$$= \langle 0.284, 0.716 \rangle$$

General Inference Procedure

- X - Query Variable
- E - Evidence Variables
- Y - Remaining Unobserved Values

$$P(X|e)$$

General Inference Procedure

- X - Query Variable
- E - Evidence Variables
- Y - Remaining Unobserved Values

$$P(X|e) = \alpha P(X, e)$$

General Inference Procedure

- X - Query Variable
- E - Evidence Variables
- Y - Remaining Unobserved Values

$$
\begin{aligned}
P(X|e) &= \alpha P(X, e) \\
&= \alpha \sum_{y} P(X, e, y)
\end{aligned}
$$

$$
\begin{aligned}
&P(B|j,m) \\
&= \alpha P(B,j,m) \\
&= \alpha \sum_e \sum_a P(B,e,a,j,m) \\
&= \alpha \sum_e \sum_a P(B)P(e)P(a|B,e)P(j|a)P(m|a) \\
&= \alpha P(B) \sum_e P(e) \sum_a P(a|B,e)P(j|a)P(m|a)
\end{aligned}
$$

$$
\begin{aligned}
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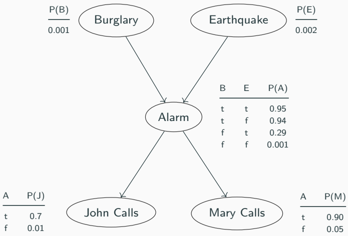
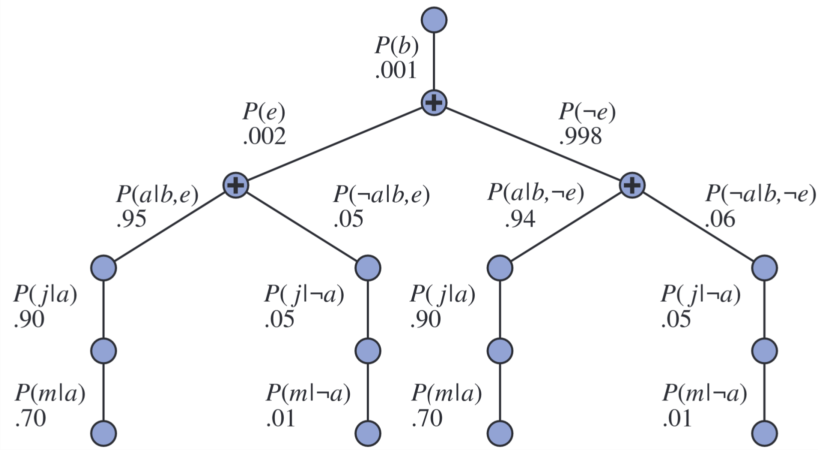
\end{aligned}
$$

$$P(b)\sum_{e} P(e)\sum_{a} P(a|b,e)P(j|a)P(m|a)$$
$$= 0.001 * (0.002 * (0.95 * 0.9 * 0.7 + 0.05 * 0.05 * 0.01)$$
$$+ 0.998 * (0.94 * 0.90 * 0.7 + 0.06 * 0.05 * 0.01))$$
$$= 0.00059224259$$




P(b)
.001
P(e)
.002
P(¬e)
.998
P(a|b,e)
.95
P(¬a|b,e)
.05
P(a|b,¬e)
.94
P(¬a|b,¬e)
.06
P(j|a)
.90
P(j|¬a)
.05
P(j|a)
.90
P(j|¬a)
.05
P(m|a)
.70
P(m|¬a)
.01
P(m|a)
.70
P(m|¬a)
.01

Variable Elimination Algorithm

$$P(B|j,m) = \alpha P(B) \sum_{e} P(e) \sum_{a} P(a|B,e) P(j|a) P(m|a)$$

Variable Elimination Algorithm

$$P(B|j,m) = \alpha \underbrace{P(B)}_{f_1(B)} \sum_e \underbrace{P(e)}_{f_2(E)} \sum_a \underbrace{P(a|B,e)}_{f_3(A,B,E)} \underbrace{P(j|a)}_{f_4(A)} \underbrace{P(m|a)}_{f_5(A)}$$

Variable Elimination Algorithm

$$P(B|j,m) = \alpha \underbrace{P(B)}_{f_1(B)} \sum_e \underbrace{P(e)}_{f_2(E)} \sum_a \underbrace{P(a|B,e)}_{f_3(A,B,E)} \underbrace{P(j|a)}_{f_4(A)} \underbrace{P(m|a)}_{f_5(A)}$$

$$f_5(A) = \langle P(m|a), P(m|\neg a) \rangle$$

Variable Elimination Algorithm

$$P(B|j,m) = \alpha f_1(B) \times \sum_e f_2(E) \times \sum_a f_3(A,B,E) \times f_4(A) \times f_5(A)$$

Variable Elimination Algorithm

$$P(B|j,m) = \alpha f_1(B) \times \sum_e f_2(E) \times \sum_a f_3(A,B,E) \times f_4(A) \times f_5(A)$$

Variable Elimination Algorithm

$$P(B|j,m) = \alpha f_1(B) \times \sum_e f_2(E) \times \sum_a f_3(A,B,E) \times f_4(A) \times f_5(A)$$

$$f_6(B,E) = \sum_a f_3(A,B,E) \times f_4(A) \times f_5(A)$$

Variable Elimination Algorithm

$$P(B|j,m) = \alpha f_1(B) \times \sum_e f_2(E) \times \sum_a f_3(A,B,E) \times f_4(A) \times f_5(A)$$

$$f_6(B,E) = \sum_a f_3(A,B,E) \times f_4(A) \times f_5(A)$$

$$= (f_3(a,B,E) \times f_4(a) \times f_5(a)) +$$

$$(f_3(\neg a,B,E) \times f_4(\neg a) \times f_5(\neg a))$$

Variable Elimination Algorithm

$$P(B|j,m) = \alpha f_1(B) \times \sum_e f_2(E) \times f_6(B,E)$$

Variable Elimination Algorithm

$$P(B|j, m) = \alpha f_1(B) \times \sum_e f_2(E) \times f_6(B, E)$$

Variable Elimination Algorithm

$$P(B|j,m) = \alpha f_1(B) \times \sum_e f_2(E) \times f_6(B,E)$$

$$f_7(B) = \sum_e f_2(E) \times f_6(B,E)$$

Variable Elimination Algorithm

$$P(B|j,m) = \alpha f_1(B) \times \sum_e f_2(E) \times f_6(B,E)$$

$$f_7(B) = \sum_e f_2(E) \times f_6(B,E)$$

$$= (f_2(e) \times f_6(B,e)) +$$

$$(f_2(\neg e) \times f_6(B,\neg e))$$

$$P(B|j, m) = \alpha f_1(B) \times f_7(B)$$

Variable Elimination Algorithm

Pointwise Product

$$f_1(A,B) \times f_2(B,C) = f(A,B,C)$$

A	B	$f_1(A,B)$	B	C	$f_2(B,C)$
T	T	0.3	T	T	0.2
T	F	0.7	T	F	0.8
F	T	0.9	T	T	0.6
F	F	0.1	F	F	0.4

A	B	C	$f(A,B,C)$
T	T	T	0.3×0.2
T	T	F	0.3×0.8
T	F	T	0.7×0.6
T	F	F	0.7×0.4
F	T	T	0.9×0.2
F	T	F	0.9×0.8
F	F	T	0.1×0.6
F	F	F	0.1×0.4

Variable Elimination Algorithm

Summing Out

$$\sum_{e} f_2(E) \times f_3(A, B, E) \times f_4(A) \times f_5(A)$$

Variable Elimination Algorithm

Summing Out

$$\sum_e f_2(E) \times f_3(A, B, E) \times f_4(A) \times f_5(A)$$
$$= f_4(A) \times f_5(A) \times \sum_e f_2(E) \times f_3(A, B, E)$$

Variable Elimination Example

$$
\begin{aligned}
P(J|b) &= \alpha \sum_e \sum_a \sum_m P(J, b, e, a, m) \\
&= \alpha \sum_e \sum_a \sum_m P(b)P(e)P(a|b,e)P(J|a)P(m|a) \\
&= \alpha' \sum_e P(e) \sum_a P(a|b,e)P(J|a) \sum_m P(m|a) \\
&= \alpha' \sum_e f_1(E) \times \sum_a f_2(A,E) \times f_3(J,A) \\
&= \alpha' \sum_e f_1(E) \times f_4(J,E) \\
&= \alpha' f_5(J)
\end{aligned}
$$

Variable Elimination Example

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&= \alpha \sum_e \sum_a \sum_m P(b)P(e)P(a|b,e)P(J|a)P(m|a) \\
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&= \alpha \sum_e \sum_a \sum_m P(b)P(e)P(a|b,e)P(J|a)P(m|a) \\
&= \alpha' \sum_e \underbrace{P(e)}_{f_1(E)} \sum_a \underbrace{P(a|b,e)}_{f_2(A,E)} \underbrace{P(J|a)}_{f_3(J,A)} \underbrace{\sum_m P(m|a)}_{=1} \\
&= \alpha' \sum_e f_1(E) \times \sum_a f_2(A,E) \times f_3(J,A) \\
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\end{aligned}
$$

Variable Ordering

$$P(B|j,m) = \alpha f_1(B) \times \sum_e f_2(E) \sum_a f_3(A,B,E) \times f_4(A) \times f_5(A)$$

Variable Ordering

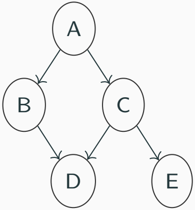
$$P(B|j,m) = \alpha f_1(B) \times \sum_e f_2(E) \sum_a f_3(A,B,E) \times f_4(A) \times f_5(A)$$

Variable Ordering

$$P(B|j,m) = \alpha f_1(B) \times \sum_e f_2(E) \times \sum_a f_3(A,B,E) \times f_4(A) \times f_5(A)$$

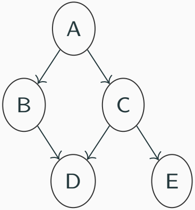
Bonus: Choosing the Order of Elimination

$$P(E) = \alpha \sum_{a,b,c,d} P(a)P(b|a)P(c|a)P(d|b,c)P(E|c)$$



Bonus: Choosing the Order of Elimination

$$P(E) = \alpha \sum_{a,b,c,d} f_1(A) \times f_2(A,B) \times f_3(A,C) \times f_4(D,B,C) \times f_5(C,E)$$



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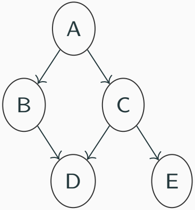

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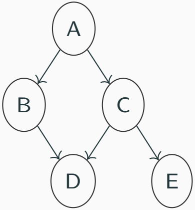

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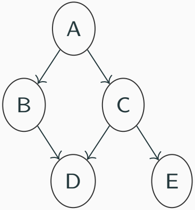

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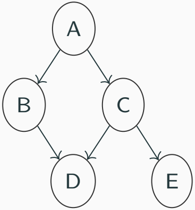

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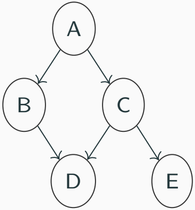

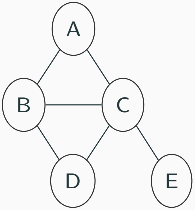

Summary

- Exact Inference in Bayesian Networks
- Inference by Enumeration
- Variable Elimination algorithm