1 Measurement-Based Quantum Computing:

What, Why & How

2 The $J(\theta)$ quantum gate

3 MBQC as Universal Model of Quantum Computation
Measurement-Based Quantum Computing:

What, Why & How
**MBQC: What (Model of Quantum Computation)**

**Circuit.** Basic mechanism:

- Evolve unitarily qubits through a circuit by applying on the qubits the gates one-by-one
- Measure (read-out) at the end to convert quantum information to classical
- Resource Cost: **Number of Gates**
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**MBQC.** (also known as **one-way quantum computer**).

Basic mechanism:

- Start with a large (generic) entangled state consisting of multiple qubits
- Make single-qubit measurements in suitably chosen bases (depending on the computation).
  Single-qubit measurements are easy to perform.
- Resource Cost: **Entanglement “consumed”**
For certain quantum hardware and architectures is easier to implement (e.g. photonic)

Has alternative ways to treat fault-tolerance and error correction (potentially advantageous)

Certain applications are easier in MBQC (see later Lecture for crypto related)

Foundationally a different perspective (e.g. the role of contextuality or certain complexity theoretic implications can be better seen in MBQC).
Gate Teleportation.

1. Entangle unknown qubit with a fixed qubit
2. Measure the unknown qubit
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Result: The information of the unknown state is “teleported” to the second qubit with an extra gate applied (see later).
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General MBQC “Ingredients”:

1. Large entangled quantum state with many qubits (resource state) – “consumed” during the computation. Easy to prepare and same for different computations.
2. Perform computation by single qubit measurements (easy to implement).
What resource state and what measurements are needed to implement a universal set of gates?

How to combine those gates for universal computation?

Does the order of measurements matters? Can we parallelise some of these measurements?

How to include an (unknown) quantum state $|\psi_0\rangle$ as input?

(Ans: Entangle this state at one side of the resource. Then measure all qubits, one-by-one.)
A: Resource States

- Entangled states used are called **graph states**.
- Given graph $G = (V, E)$ with vertices $V$ and edges $E$

\[
\text{Resulting state: } |G\rangle = Q(a, b) \in E \land Z(a, b) |+\rangle \otimes V
\]

Note: $\land Z$'s commute, so order does not matter.
A: Resource States

- Entangled states used are called graph states.
- Given graph $G = (V, E)$ with vertices $V$ and edges $E$
  
  - Place at each vertex a qubit at $|+\rangle$
  - For each edge apply $\land Z$ to entangle the vertices
  
  Resulting state: $|G\rangle = \prod_{(a,b) \in E} \land Z^{(a,b)} |+\rangle \otimes V$

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A: Resource States

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Remarks:

- If the graph used is subset of $d$-dimensional lattice the state are also known as **cluster states**.

- Graph states are highly entangled between all qubits. Entanglement remains after measuring some qubits

- Entanglement is “consumed” during the computation \(\Rightarrow\) **resource** of the computation.
Remarks:

- If the graph used is subset of $d$-dimensional lattice the state are also known as **cluster states**.

- Graph states are highly entangled between **all** qubits. Entanglement remains after measuring some qubits.

- Entanglement is “consumed” during the computation $\Rightarrow$ **resource** of the computation.

  Is called **one-way quantum computation**, since the resource is consumed during the computation $\Rightarrow$ non-reversible.
B: Measurements

- **Single-qubit** measurements
  - **Subscript** denotes qubit measured
  - **Superscript** denotes basis of measurement

- **Bases used:**
  \[ M_j^\theta = \{ |+\theta\rangle, |-\theta\rangle \} \] for all \( \theta \) and \[ M_j^Z = \{ |0\rangle, |1\rangle \} \]

Recall that |±\theta⟩ = \( \frac{1}{\sqrt{2}} (|0\rangle ± e^{i\theta} |1\rangle) \)

The role of the \( Z \) measurement will be explained later

- Measurements have binary outcome, for qubit \( j \) we denote the one outcome \( s_j = 0 \) and the second \( s_j = 1 \)
Measurement outcomes are random. To achieve deterministic outcome (unitary), we need to adapt the measurement angles to “cancel” the randomness of previous measurements.

The (partial) order of measurements and adaptivity will be explored in the next lecture.

Here we see how to obtain in MBQC the “$J(\theta)$” universal gate-set, up to certain “corrections”
The $|\pm\theta\rangle$-basis. For all $\theta$ we define:

$$|+\theta\rangle = \frac{1}{\sqrt{2}} \left( |0\rangle + e^{i\theta} |1\rangle \right), \quad |-\theta\rangle = \frac{1}{\sqrt{2}} \left( |0\rangle - e^{i\theta} |1\rangle \right)$$

Note: $\{ |+\theta\rangle, |-\theta\rangle \}$ is a basis and $\theta = 0$ is the $|\pm\rangle$-basis.

$$|0\rangle = \frac{1}{\sqrt{2}} \left( |+\theta\rangle + |-\theta\rangle \right), \quad |1\rangle = \frac{1}{\sqrt{2}} e^{-i\theta} \left( |+\theta\rangle - |-\theta\rangle \right)$$
The \( J(\theta) \) universal gate-set:

The set of quantum gates \( \{\land Z, J(\theta) \text{ for all } \theta\} \) is universal
The $J(\theta)$ universal gate-set:

The set of quantum gates $\{\text{\textsc{w}}Z, J(\theta) \text{ for all } \theta\}$ is universal.

Recall: $R(\theta) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix}$

We define the Hadamard rotated phase gate:

$$J(\theta) = H R(\theta) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & e^{i\theta} \\ 1 & -e^{i\theta} \end{bmatrix}$$
The $J(\theta)$ universal gate-set:

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We define the Hadamard rotated phase gate:

$$J(\theta) = HR(\theta) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & e^{i\theta} \\ 1 & -e^{i\theta} \end{bmatrix}$$

- Any single-qubit unitary gate can be decomposed as:

$$U = J(0)J(\theta_1)J(\theta_2)J(\theta_3)$$

for some $\theta_1, \theta_2, \theta_3$.

- For universal set we need a two-qubit gate: $\land Z$
The $|H\rangle$ maximally entangled state:

Recall the controlled-Z gate ($\wedge Z$)

$$\wedge Z |i\rangle |j\rangle = (-1)^{ij} |i\rangle |j\rangle$$

is symmetric w.r.t. inputs (unlike $\wedge X |i\rangle |j\rangle = |i\rangle |i \oplus j\rangle$)

We define:

$$|H\rangle := \wedge Z |+\rangle \otimes |+\rangle = \frac{1}{2} (|00\rangle + |01\rangle + |10\rangle - |11\rangle)$$

This state is maximally entangled:

$$|H\rangle = \frac{1}{\sqrt{2}} (|0\rangle |+\rangle + |1\rangle |-\rangle) = \frac{1}{\sqrt{2}} (|+\rangle |0\rangle + |\rangle |1\rangle)$$

$$|H\rangle = (I \otimes H) |\Phi^+\rangle = (H \otimes I) |\Phi^+\rangle = \wedge Z |+\rangle \otimes |+\rangle$$

Note1: $\wedge Z$ acts on $|+\rangle$’s entangles qubits symmetrically

Note2: The $|H\rangle$ is a two-qubit state not to be confused with the Hadamard operator $H$. 
The $J(\theta)$ quantum gate
It is called “Measurement Pattern”

- **Resource State:**
  - A graph with labelled vertices (qubits)
  - Set of vertices that are **input** and **output** of the computation

Unless stated otherwise: inputs are on the left-hand side; outputs are on the right-hand side (and are not-measured)

- **Measurements:**

Angles that each qubit is measured are denoted on the vertex

In general, angles need to be adaptively corrected. Denoted angles are the “default” un-corrected ones (see next lecture)
**Gate Teleportation:** We start with unknown state $|\psi\rangle_1 = a|0\rangle_1 + b|1\rangle_1$ plugged in the following MBQC pattern:

The total state after entangling ($\wedge Z$) becomes:

$$|\phi\rangle_{12} := \wedge Z_{12} (|\psi\rangle_1 \otimes |+\rangle_2) = a|0+\rangle_{12} + b|1-\rangle_{12}$$
The $J(\theta)$ single-qubit gate

To see the effect of the measurement $M_1^\theta$, we express qubit 1 (that is to be measured) in the corresponding $|\pm\theta\rangle$ basis (see expansion of $|0\rangle, |1\rangle$ in this basis):

$$|\phi\rangle_{12} = \frac{a}{\sqrt{2}} (|+\theta\rangle_1 + |--\theta\rangle_1) |+\rangle_2 + \frac{b}{\sqrt{2}} e^{-i\theta} (|+\theta\rangle_1 - |--\theta\rangle_1) |--\rangle_2$$

$$= \frac{1}{\sqrt{2}} |+\theta\rangle_1 (a |+\rangle_2 + be^{-i\theta} |--\rangle_2) + \frac{1}{\sqrt{2}} |--\theta\rangle_1 (a |+\rangle_2 - be^{-i\theta} |--\rangle_2)$$

$$= \frac{1}{\sqrt{2}} |+\theta\rangle_1 (a |+\rangle_2 + be^{-i\theta} |--\rangle_2) + \frac{1}{\sqrt{2}} |--\theta\rangle_1 (a |+\rangle_2 - be^{-i\theta} |--\rangle_2)$$

(1)

We can re-express now the state of qubit 2 in each of the two terms in the RHS of Eq 1.
The $J(\theta)$ single-qubit gate

- We note that the first term can be written as:

$$HR(-\theta)(a |0\rangle + b |1\rangle) = H \left( a |0\rangle + be^{-i\theta} |1\rangle \right) = a |+\rangle + be^{-i\theta} |-\rangle$$

- and that the second term can be written as:

$$XHR(-\theta)(a |0\rangle + b |1\rangle) = Xa |+\rangle + Xbe^{-i\theta} |-\rangle = a |+\rangle - be^{-i\theta} |-\rangle$$

- We therefore have:

$$|\phi\rangle_{12} = |+\rangle_1 (X_2)^0 J(-\theta)_2 |\psi\rangle_2 + |-\rangle_1 (X_2)^1 J(-\theta)_2 |\psi\rangle_2$$
We can see that measuring qubit 1 in the $M_1^\theta$-basis we end-up with qubit 2 being at the state $X^{s_1} J(-\theta) |\psi\rangle$, where $s_1$ is the outcome of qubit’s 1 measurement.

**Interpretation:** We have teleported the state $|\psi\rangle_1$ to qubit 2, and in the same time we have applied on it, the gate $J(-\theta)$ along with an extra operation $X^{s_1}$ that depends on the previous measurement outcome.

To restore “determinism” we need to “cancel” the gate $X^{s_1}$, something that is possible by adapting the measurement angles (**see next lecture**).
The $J(\theta)$ single-qubit gate: Summary

The above measurement pattern results to:

$$X^{s_1} J(-\theta) |\psi\rangle_2 = X^{s_1} H R(-\theta) |\psi\rangle_2$$

Examples:

- $\theta = 0$: Output $X^{s_1} H |\psi\rangle_2$
- $\theta = \pi$: Output $X^{s_1} H Z |\psi\rangle_2$
- $\theta = \pi/2$: Output $X^{s_1} H R(-\pi/2) |\psi\rangle_2 = X^{s_1} H \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} |\psi\rangle_2$
MBQC as Universal Model of Quantum Computation
How to apply consecutively two $J(\theta)$-gates:

The $(J(\theta_2)J(\theta_1))$-gate MBQC pattern

Operators (or measurements) acting on different subsystems commute (can be performed in arbitrary order)
How to apply consecutively two \( J(\theta) \)-gates:

Operators (or measurements) acting on different subsystems commute (can be performed in arbitrary order). We can break the pattern of the figure to two steps:

1. Consider **qubit 1** and **qubit 2** alone: Prepare these qubits, entangle them and measure **qubit 1** (see previous example).
How to apply consecutively two $J(\theta)$-gates:

| $s_1\rangle_1$ | $\theta_2, s_2$ | $X^s J(\theta_1) |\psi\rangle_2$ | $|+\rangle_3$ |

Step 2: after measuring qubit 1

Operators (or measurements) acting on different subsystems commute (can be performed in arbitrary order)

We can break the pattern of the figure to two steps:

1. Consider qubit 1 and qubit 2 alone: Prepare these qubits, entangle them and measure qubit 1 (see previous example).
2. Then prepare qubit 3 entangle qubit 2 with qubit 3 and measure qubit 2.

Step 2 is again the $J(\theta)$-gate but has as input qubit 2 in the state produced in step 1.
Composing single-qubit measurement patterns

In more details the two steps:

1. The input was $|\psi\rangle_1$, measurement angle $\theta_1$ and outcome $s_1$:

$$|s_1\rangle \otimes X_2^{s_1} H_2 R_2(-\theta_1) |\psi\rangle_2 = |s_1\rangle \otimes X_2^{s_1} J_2(-\theta_1) |\psi\rangle_2$$

2. The input was $X_2^{s_1} J_2(-\theta_1) |\psi\rangle_2$ (we can ignore qubit 1 now that is measured), measurement angle $\theta_2$ and outcome $s_2$:

$$|s_2\rangle \otimes X_3^{s_2} J_3(-\theta_2)(X_3^{s_1} J_3(-\theta_1) |\psi\rangle_3$$
Composing single-qubit measurement patterns

In more details the two steps:

1. The input was $|\psi\rangle_1$, measurement angle $\theta_1$ and outcome $s_1$:

   $$|s_1\rangle \otimes X_{s_1}^2 H_2 R_2(-\theta_1) |\psi\rangle_2 = |s_1\rangle \otimes X_{s_1}^2 J_2(-\theta_1) |\psi\rangle_2$$

2. The input was $X_{s_1}^2 J_2(-\theta_1) |\psi\rangle_2$ (we can ignore qubit 1 now that is measured), measurement angle $\theta_2$ and outcome $s_2$:

   $$|s_2\rangle \otimes X_{s_2}^3 J_3(-\theta_2)(X_{s_1}^3 J_3(-\theta_1) |\psi\rangle_3)$$

The “corrections” $X_{s_1}^2, X_{s_2}^3$ will be dealt at next lecture in the general case.

Now note that the output (qubit 3) is now at the state $|\psi\rangle$ with the gates $J(-\theta_2)J(-\theta_1)$ applied.
Composing single-qubit measurement patterns

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Now note that the output (qubit 3) is now at the state $|\psi\rangle$ with the gates $J(-\theta_2)J(-\theta_1)$ applied.

- (Standard) Teleportation: Case $\theta_1 = \theta_2 = 0$:

$$X^{s_2}_3 H_3 X^{s_1}_3 H_3 |\psi\rangle_3 = X^{s_2}_3 Z^{s_1}_3 H_3 H_3 |\psi\rangle = X^{s_2}_3 Z^{s_1}_3 |\psi\rangle$$
Any single-qubit gate can be implemented repeating the $J(\theta)$ pattern since any single-qubit unitary can be written using three angles $\theta_1, \theta_2, \theta_3$.

$$U = J(0)J(-\theta_3)J(-\theta_2)J(-\theta_1)$$

This pattern results to (step by step):

$$|s_1\rangle \otimes X^{s_1} J(-\theta_1) |\psi\rangle$$
Any single-qubit gate can be implemented repeating the $J(\theta)$ pattern since any single-qubit unitary can be written using three angles $\theta_1, \theta_2, \theta_3$.

$$U = J(0)J(-\theta_3)J(-\theta_2)J(-\theta_1)$$

A general single-qubit gate: MBQC pattern

This pattern results to (step by step):

$$|s_1\rangle \otimes |s_2\rangle \otimes X^{s_2} J(-\theta_2) X^{s_1} J(-\theta_1) |\psi\rangle$$
**Any single-qubit gate** can be implemented repeating the $J(\theta)$ pattern since any single-qubit unitary can be written using three angles $\theta_1, \theta_2, \theta_3$

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A general single-qubit gate: MBQC pattern

- This pattern results to (step by step):

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**General Single-Qubit Gate**

Any single-qubit gate can be implemented repeating the $J(\theta)$ pattern since any single-qubit unitary can be written using three angles $\theta_1, \theta_2, \theta_3$

$$U = J(0)J(-\theta_3)J(-\theta_2)J(-\theta_1)$$

A general single-qubit gate: MBQC pattern

- This pattern results to (step by step):

$$|s_1\rangle \otimes |s_2\rangle \otimes |s_3\rangle \otimes |s_4\rangle \otimes X^{s_4} J(0) X^{s_3} J(-\theta_3) X^{s_2} J(-\theta_2) X^{s_1} J(-\theta_1) |\psi\rangle$$
What is missing to achieve the universal $J(\theta)$ gate-set is a way to implement the $\wedge Z$-gate.

We already have the $\wedge Z$-gate in our generating graph process.

Care is needed, as it should be applied to qubits not already measured (2-dim measurement pattern).
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Entangling should happen without obstructing the “flow” (teleportation path).
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Information “flows” as qubits are teleported through the measurement pattern.

Entangling should happen without obstructing the “flow” (teleportation path).

Horizontal $\wedge Z$ is used to teleport information (and gates).

Vertical $\wedge Z$ is used as the 2-qubit gate.
We will ignore the corrections (assume all $s_i$’s are zero).
We will see step by step the pattern:

The ($\wedge Z_{12}) (H_1 \otimes H_2$)-gate MBQC pattern
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The $(\land Z_{12})(H_1 \otimes H_2)$-gate MBQC pattern
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\[
\begin{align*}
|s_1\rangle_1 & = 0 \\
H|\psi\rangle_3 \\
|s_2\rangle_2 & = 0 \\
H|\phi\rangle_4
\end{align*}
\]

The $(\wedge Z_{12})(H_1 \otimes H_2)$-gate MBQC pattern
Example: details

1. Consider qubit 1 and qubit 3 (the effect of measuring qubit 1).

\[ |s_1\rangle_1 \otimes X_{s_1}^3 H_3 |\psi\rangle_3 \]

2. Consider qubit 2 and qubit 4 (the effect of measuring qubit 1).

\[ |s_2\rangle_2 \otimes X_{s_2}^4 H_4 |\phi\rangle_4 \]

3. We apply a $\wedge Z$ on the qubits 3 and 4.

\[ |s_1\rangle_1 \otimes |s_2\rangle_2 \otimes \wedge Z_{34} (X_{s_1}^3 H_3 |\psi\rangle_3 \otimes X_{s_2}^4 H_4 |\phi\rangle_4) \]

The corrections ($X$'s that depend on measurement outcomes) will be formally treated later (L16). The net effect (baring corrections – setting $s_i = 0$) is:

\[ (\wedge Z) (H \otimes H) (|\psi\rangle_3 \otimes |\phi\rangle_4) \]
Consider qubit 1 and qubit 3 (the effect of measuring qubit 1).

\[ |s_1\rangle_1 \otimes X^{s_1}_3 H_3 |\psi\rangle_3 \]

Consider qubit 2 and qubit 4 (the effect of measuring qubit 1).

\[ |s_2\rangle_2 \otimes X^{s_2}_4 H_4 |\phi\rangle_4 \]

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- The corrections ($X$’s that depend on measurement outcomes) will be formally treated later (L16)

- The net effect (barring corrections – setting $s_i = 0$) is:

\[ (\wedge Z) (H \otimes H) (|\psi\rangle \otimes |\phi\rangle) \]
We have measurement pattern for the $J(-\theta)$ gate:

Input qubit entangled with another one and measured in the $M^\theta$ basis.
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General single qubit gates can be obtained by combining the previous gates and noting that all single qubit unitaries can be decomposed as $J(0)J(\theta_1)J(\theta_2)J(\theta_3)$
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For two-qubit gate, we use $\wedge Z$ that is already in the graph state, but:

Need to apply it “vertically” to avoid obstructing the “flow” of the computation
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**Next Lecture:** formally how to treat “corrections” and resort deterministic application of gates!
Further Reading


