

Introduction to Quantum Computing

Lecture 17: Measurement-Based Quantum Computing (MBQC) I

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① Measurement-Based Quantum Computing:

What, Why & How

② The $J(\theta)$ quantum gate

③ MBQC as Universal Model of Quantum Computation

Measurement-Based Quantum Computing:

What, Why & How

MBQC: What (Model of Quantum Computation)

Circuit. Basic mechanism:

- Evolve unitarily qubits through a circuit by applying on the qubits the gates one-by-one
- Measure (read-out) at the end to convert quantum information to classical
- Resource Cost: **Number of Gates**

MBQC: What (Model of Quantum Computation)

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MBQC. (also known as **one-way quantum computer**).

Basic mechanism:

- Start with a large (generic) entangled state consisting of multiple qubits
- Make single-qubit measurements in suitably chosen bases (depending on the computation).
Single-qubit measurements are easy to perform.
- Resource Cost: **Entanglement “consumed”**

- For **certain quantum hardware and architectures** is easier to implement (e.g. photonic)
- Has alternative **ways to treat fault-tolerance** and error correction (potentially advantageous)
- Certain **applications** are easier in MBQC (see later Lecture for crypto related)
- **Foundationally** a different perspective (e.g. the role of contextuality or certain complexity theoretic implications can be better seen in MBQC).

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General MBQC “Ingredients”:

- 1 Large entangled quantum state with many qubits (**resource state**) – “consumed” during the computation.
Easy to prepare and same for different computations.
- 2 Perform computation by **single qubit measurements** (easy to implement).

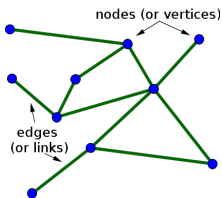
MBQC: What do we need?

- What **resource state** and what **measurements** are needed to implement a universal set of gates?
- How to combine those gates for **universal computation**?
- Does the order of measurements matters? Can we parallelise some of these measurements?
- How to include an (unknown) quantum state $|\psi_0\rangle$ as input?

(Ans: Entangle this state at one side of the resource. Then measure **all** qubits, one-by-one.)

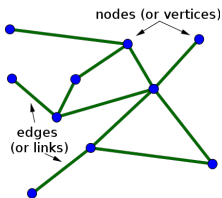
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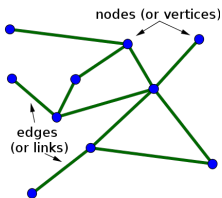


- Place at each vertex a qubit at $|+\rangle$
- For each edge apply $\wedge Z$ to entangle the vertices

Resulting state: $|G\rangle = \prod_{(a,b) \in E} \wedge Z^{(a,b)} |+\rangle^{\otimes V}$

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Note: $\wedge Z$'s commute, so order does not matter

Remarks:

- If the graph used is subset of d -dimensional lattice the state are also known as **cluster states**.
- Graph states are highly entangled between **all** qubits.
Entanglement remains after measuring some qubits
- Entanglement is “consumed” during the computation \Rightarrow **resource** of the computation.

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- If the graph used is subset of d -dimensional lattice the state are also known as **cluster states**.
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Is called **one-way quantum computation**, since the resource is consumed during the computation \Rightarrow non-reversible.

B: Measurements

- **Single-qubit** measurements
Subscript denotes qubit measured
Superscript denotes basis of measurement

- Bases used:

$$M_j^\theta = \{ |+\theta\rangle, |-\theta\rangle \} \text{ for all } \theta \text{ and } M_j^Z = \{ |0\rangle, |1\rangle \}$$

$$\text{Recall that } |\pm\theta\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm e^{i\theta}|1\rangle)$$

The role of the Z measurement will be explained later

- Measurements have binary outcome, for qubit j we denote the one outcome $s_j = 0$ and the second $s_j = 1$

Resource States and Measurements

- Measurement outcomes are random. To achieve deterministic outcome (unitary), we need to adapt the measurement angles to “cancel” the randomness of previous measurements.
- The (partial) **order** of measurements and **adaptivity** will be explored in the next lecture.
- Here we see how to obtain in MBQC the “ $J(\theta)$ ” **universal gate-set**, up to certain “corrections”

Some Useful Background

- 1 **The $|\pm\theta\rangle$ -basis.** For all θ we define:

$$|+\theta\rangle = \frac{1}{\sqrt{2}} (|0\rangle + e^{i\theta} |1\rangle), \quad |-\theta\rangle = \frac{1}{\sqrt{2}} (|0\rangle - e^{i\theta} |1\rangle)$$

Note: $\{|+\theta\rangle, |-\theta\rangle\}$ is a basis and $\theta = 0$ is the $|\pm\rangle$ -basis.

$$|0\rangle = \frac{1}{\sqrt{2}} (|+\theta\rangle + |-\theta\rangle), \quad |1\rangle = \frac{1}{\sqrt{2}} e^{-i\theta} (|+\theta\rangle - |-\theta\rangle)$$

2 The $J(\theta)$ universal gate-set:

The set of quantum gates $\{\wedge Z, J(\theta) \text{ for all } \theta\}$ is universal

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Recall: $R(\theta) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix}$

We define the Hadamard rotated phase gate:

$$J(\theta) = HR(\theta) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & e^{i\theta} \\ 1 & -e^{i\theta} \end{bmatrix}$$

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- Any single-qubit unitary gate can be decomposed as:

$$U = J(0)J(\theta_1)J(\theta_2)J(\theta_3)$$

for some $\theta_1, \theta_2, \theta_3$

- For universal set we need a two-qubit gate: $\wedge Z$

Some Useful Background

3 The $|H\rangle$ maximally entangled state:

Recall the controlled-Z gate ($\wedge Z$)

$$\wedge Z |i\rangle |j\rangle = (-1)^{ij} |i\rangle |j\rangle$$

is symmetric w.r.t. inputs (unlike $\wedge X |i\rangle |j\rangle = |i\rangle |i \oplus j\rangle$)

We define:

$$|H\rangle := \wedge Z |+\rangle \otimes |+\rangle = \frac{1}{2} (|00\rangle + |01\rangle + |10\rangle - |11\rangle)$$

This state is maximally entangled:

$$|H\rangle = \frac{1}{\sqrt{2}} (|0\rangle |+\rangle + |1\rangle |-\rangle) = \frac{1}{\sqrt{2}} (|+\rangle |0\rangle + |-\rangle |1\rangle)$$

$$|H\rangle = (\mathbb{I} \otimes H) |\Phi^+\rangle = (H \otimes \mathbb{I}) |\Phi^+\rangle = \wedge Z |+\rangle \otimes |+\rangle$$

Note1: $\wedge Z$ acts on $|+\rangle$'s entangles qubits symmetrically

Note2: The $|H\rangle$ is a two-qubit state not to be confused with the Hadamard operator H .

The $J(\theta)$ quantum gate

It is called “**Measurement Pattern**”

- **Resource State:**

- A graph with labelled vertices (qubits)
- Set of vertices that are **input** and **output** of the computation

Unless stated otherwise: inputs are on the left-hand side; outputs are on the right-hand side (and are not-measured)

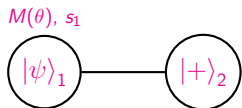
- **Measurements:**

Angles that each qubit is measured are denoted on the vertex

In general, angles need to be adaptively corrected. Denoted angles are the “default” un-corrected ones (see next lecture)

The $J(\theta)$ single-qubit gate

Gate Teleportation: We start with unknown state $|\psi\rangle_1 = a|0\rangle_1 + b|1\rangle_1$ plugged in the following MBQC pattern:



The $J(-\theta)$ -gate MBQC pattern

The total state after entangling ($\wedge Z$) becomes:

$$|\phi\rangle_{12} := \wedge Z_{12} (|\psi\rangle_1 \otimes |+\rangle_2) = a|0+\rangle_{12} + b|1-\rangle_{12}$$

The $J(\theta)$ single-qubit gate

- To see the effect of the measurement M_1^θ , we express **qubit 1** (that is to be measured) in the corresponding $|\pm\theta\rangle$ basis (see expansion of $|0\rangle, |1\rangle$ in this basis):

$$\begin{aligned} |\phi\rangle_{12} &= \frac{a}{\sqrt{2}}(|+\theta\rangle_1 + |-\theta\rangle_1)|+\rangle_2 + \frac{b}{\sqrt{2}}e^{-i\theta}(|+\theta\rangle_1 - |-\theta\rangle_1)|-\rangle_2 \\ &= \frac{1}{\sqrt{2}}|+\theta\rangle_1(a|+\rangle_2 + be^{-i\theta}|-\rangle_2) + \\ &\quad + \frac{1}{\sqrt{2}}|-\theta\rangle_1(a|+\rangle_2 - be^{-i\theta}|-\rangle_2) \end{aligned} \quad (1)$$

- We can re-express now the state of **qubit 2** in each of the two terms in the RHS of Eq 1

The $J(\theta)$ single-qubit gate

- We note that the first term can be written as:

$$HR(-\theta)(a|0\rangle + b|1\rangle) = H(a|0\rangle + be^{-i\theta}|1\rangle) = a|+\rangle + be^{-i\theta}|-\rangle$$

- and that the second term can be written as:

$$XHR(-\theta)(a|0\rangle + b|1\rangle) = Xa|+\rangle + Xbe^{-i\theta}|-\rangle = a|+\rangle - be^{-i\theta}|-\rangle$$

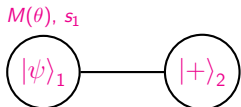
- We therefore have:

$$|\phi\rangle_{12} = |+\theta\rangle_1 (X_2)^0 J(-\theta)_2 |\psi\rangle_2 + |-\theta\rangle_1 (X_2)^1 J(-\theta)_2 |\psi\rangle_2$$

The $J(\theta)$ single-qubit gate

- We can see that measuring **qubit 1** in the M_1^θ -basis we end-up with **qubit 2** being at the state $X^{s_1} J(-\theta) |\psi\rangle$, where s_1 is the outcome of qubit's 1 measurement.
- **Interpretation:** We have teleported the state $|\psi\rangle_1$ to **qubit 2**, and in the same time we have applied on it, the gate $J(-\theta)$ along with an extra operation X^{s_1} that depends on the previous measurement outcome
- To restore “determinism” we need to “cancel” the gate X^{s_1} , something that is possible by adapting the measurement angles (**see next lecture**)

The $J(\theta)$ single-qubit gate: Summary



The $J(-\theta)$ -gate MBQC pattern

The above measurement pattern results to:

$$X^{s_1} J(-\theta) |\psi\rangle_2 = X^{s_1} H R(-\theta) |\psi\rangle_2$$

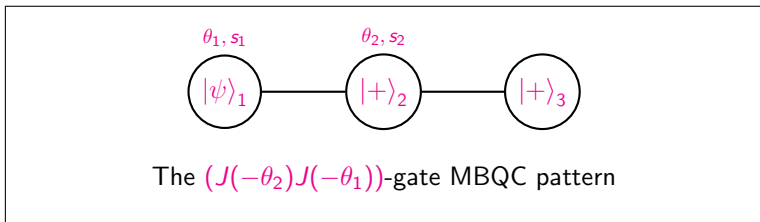
Examples:

- $\theta = 0$: Output $X^{s_1} H |\psi\rangle_2$
- $\theta = \pi$: Output $X^{s_1} H Z |\psi\rangle_2$
- $\theta = \pi/2$: Output $X^{s_1} H R(-\pi/2) |\psi\rangle_2 = X^{s_1} H \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} |\psi\rangle_2$

MBQC as Universal Model of Quantum Computation

Composing single-qubit measurement patterns

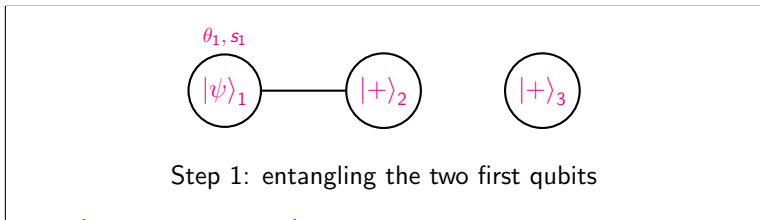
How to apply consecutively two $J(\theta)$ -gates:



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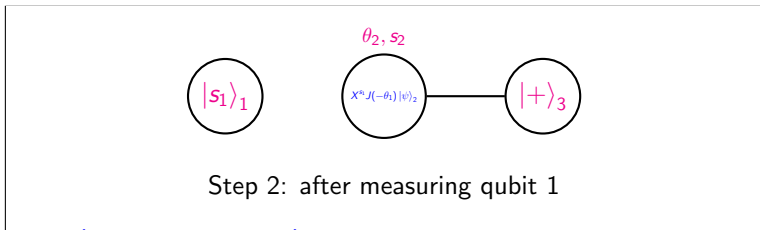
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We can break the pattern of the figure to two steps:

- 1 Consider **qubit 1** and **qubit 2** alone: Prepare these qubits, entangle them and measure **qubit 1** (see previous example)

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We can break the pattern of the figure to two steps:

- 1 Consider **qubit 1** and **qubit 2** alone: Prepare these qubits, entangle them and measure **qubit 1** (see previous example)
- 2 Then prepare **qubit 3** entangle **qubit 2** with **qubit 3** and measure **qubit 2**.

Step 2 is again the $J(-\theta)$ -gate but has as input **qubit 2** in the state produced in step 1.

Composing single-qubit measurement patterns

In more details the two steps:

- 1 The input was $|\psi\rangle_1$, measurement angle θ_1 and outcome s_1 :

$$|s_1\rangle \otimes X_2^{s_1} H_2 R_2(-\theta_1) |\psi\rangle_2 = |s_1\rangle \otimes X_2^{s_1} J_2(-\theta_1) |\psi\rangle_2$$

- 2 The input was $X_2^{s_1} J_2(-\theta_1) |\psi\rangle_2$ (we can ignore qubit 1 now that is measured), measurement angle θ_2 and outcome s_2 :

$$|s_2\rangle \otimes X_3^{s_2} J_3(-\theta_2) (X_3^{s_1} J_3(-\theta_1) |\psi\rangle_3)$$

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The “corrections” X^{s_1}, X^{s_2} will be dealt at next lecture in the general case.

Now note that the output (**qubit 3**) is now at the state $|\psi\rangle$ with the gates $J(-\theta_2)J(-\theta_1)$ applied.

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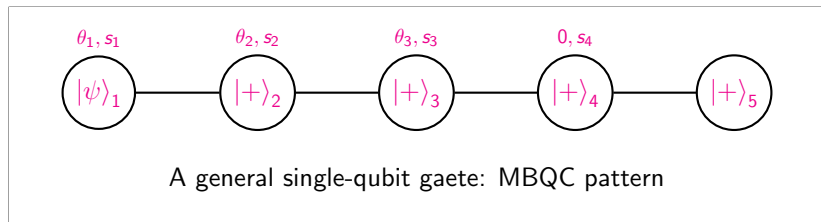
- (Standard) Teleportation: Case $\theta_1 = \theta_2 = 0$:

$$X_3^{s_2} H_3 X_3^{s_1} H_3 |\psi\rangle_3 = X_3^{s_2} Z_3^{s_1} H_3 H_3 |\psi\rangle = X_3^{s_2} Z_3^{s_1} |\psi\rangle$$

General Single-Qubit Gate

Any single-qubit gate can be implemented repeating the $J(\theta)$ pattern since any single-qubit unitary can be written using three angles $\theta_1, \theta_2, \theta_3$

$$U = J(0)J(-\theta_3)J(-\theta_2)J(-\theta_1)$$



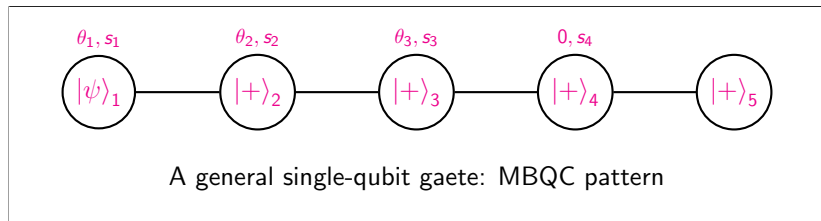
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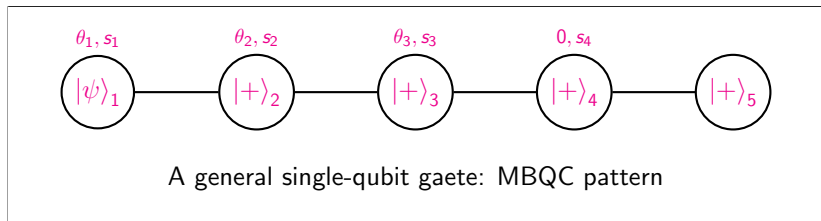
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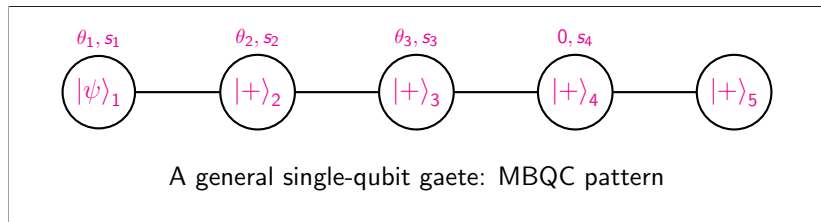
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- This pattern results to (step by step):

$$|s_1\rangle \otimes |s_2\rangle \otimes |s_3\rangle \otimes |s_4\rangle \otimes X^{s_4} J(0) X^{s_3} J(-\theta_3) X^{s_2} J(-\theta_2) X^{s_1} J(-\theta_1) |\psi\rangle$$

Two Qubit Gates

- What is missing to achieve the universal $J(\theta)$ gate-set is a way to implement the $\wedge Z$ -gate.
- We already have the $\wedge Z$ -gate in our generating graph process
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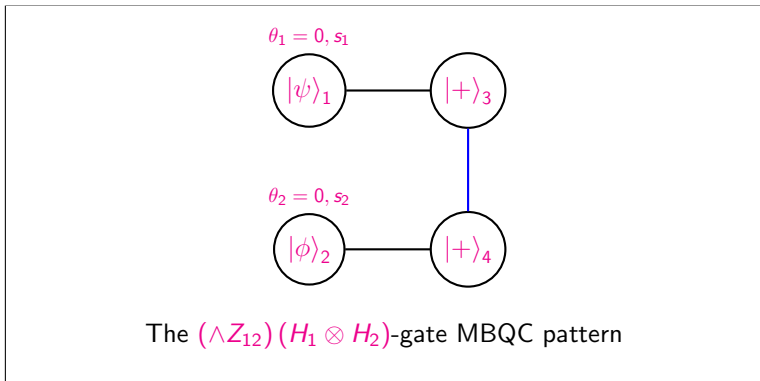
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- Information “flows” as qubits are teleported through the measurement pattern
- Entangling should happen without obstructing the “flow” (teleportation path)
- Horizontal $\wedge Z$ is used to teleport information (and gates)
- Vertical $\wedge Z$ is used as the 2-qubit gate.

Example: Two Qubit Gate

We will ignore the corrections (assume all s_i 's are zero).

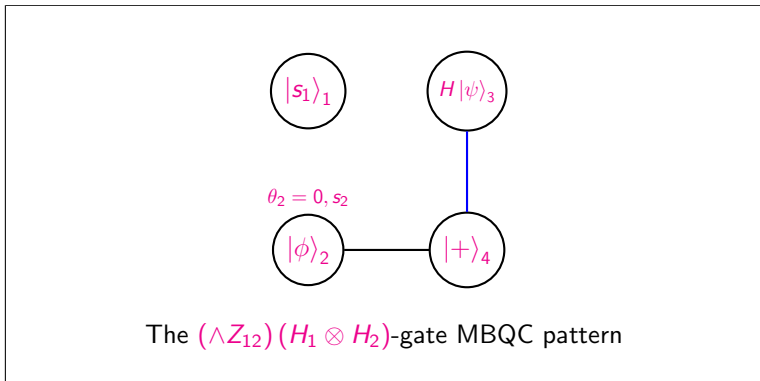
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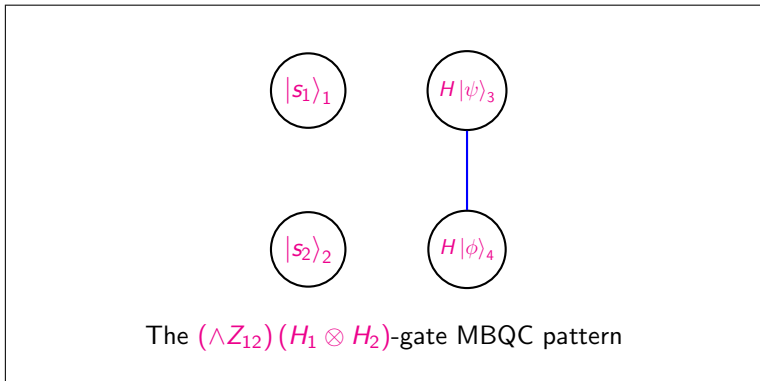
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Example: details

- 1 Consider qubit 1 and qubit 3 (the effect of measuring qubit 1).

$$|s_1\rangle_1 \otimes X_3^{s_1} H_3 |\psi\rangle_3$$

- 2 Consider qubit 2 and qubit 4 (the effect of measuring qubit 1).

$$|s_2\rangle_2 \otimes X_4^{s_2} H_4 |\phi\rangle_4$$

- 3 We apply a $\wedge Z$ on the qubits 3 and 4.

$$|s_1\rangle_1 \otimes |s_2\rangle_2 \otimes \wedge Z_{34} (X_3^{s_1} H_3 |\psi\rangle_3 \otimes X_4^{s_2} H_4 |\phi\rangle_4)$$

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- The corrections (X 's that depend on measurement outcomes) will be formally treated later (L16)
- The net effect (barring corrections – setting $s_i = 0$) is:

$$(\wedge Z) (H \otimes H) (|\psi\rangle \otimes |\phi\rangle)$$

Summary: MBQC Universality

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Input qubit entangled with another one and measured in the M^θ basis

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Need to apply it “vertically” to avoid obstructing the “flow” of the computation
- **Next Lecture:** formally how to treat “corrections” and resort deterministic application of gates!

Further Reading

- 1 One-way Quantum Computation - a tutorial introduction, D. Browne and H. Briegel, arxiv:quant-ph/0603226
- 2 An introduction to measurement based quantum computation, R. Jozsa, arxiv:quant-ph/0508124
- 3 Quantum computing with photons: introduction to the circuit model, the one-way quantum computer, and the fundamental principles of photonic experiments, S. Barz, Journal of Physics B: Atomic, Molecular and Optical Physics, Vol 48, Num. 8 (2015).
- 4 Chapter 7, Semantic Techniques in Quantum Computation – Editors Simon Gay and Ian Mackie