Introduction to Quantum Computing
Lecture 18: MBQC II

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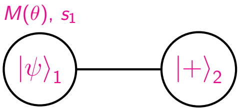
This Lecture

1. General Measurement Patterns and Background
2. How to cancel the “corrections” due to randomness
3. Output corrections, an Example and MBQC recap

Part I

General Measurement Patterns and Background

Previously in MBQC



The $J(-\theta)$-gate MBQC pattern

The above measurement pattern results to:

$$X^{s_1} J(-\theta) |\psi\rangle_2 = X^{s_1} HR(-\theta) |\psi\rangle_2$$

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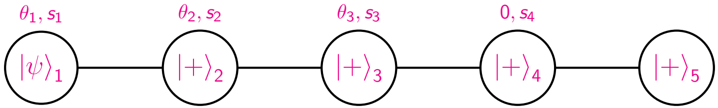
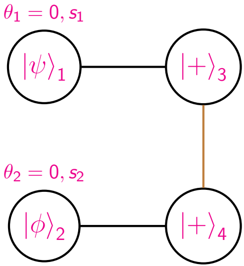


A general single-qubit gaete: MBQC pattern

The above measurement pattern results to:

$$X^{s_4} J(0) X^{s_3} J(-\theta_3) X^{s_2} J(-\theta_2) X^{s_1} J(-\theta_1) |\psi\rangle$$

Previously in MBQC



The $(\wedge Z_{12})(H_1 \otimes H_2)$-gate MBQC pattern

The above measurement pattern results to:

$$\wedge Z_{34}\,(X_3^{s_1} H_3 |\psi\rangle_3 \otimes X_4^{s_2} H_4 |\phi\rangle_4)$$

Measurement Pattern for Generic Computation

- Corrections appear when $s_i \neq 0$
- Let ϕ_i be the measurement angles that implement the desired unitary if **all** measurements give the result zero $s_i = 0 \; \forall \; i$
- The set $\{\phi_i\}_i$ determine the computation performed, and we call them **default measurement angles**.

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- **Default angles determine computation:** $\{\phi_1, \phi_2, \cdots\}$
- **Corrected angles** are the one **used for measurements:** $\{\phi'_1, \phi'_2, \cdots\}$

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Definition: An entanglement graph (G, I, O), with I, O input resp output vertices, has flow if there exists a map $f : O^c \to I^c$ and a partial order $\preceq$ over qubits

1. $x \sim f(x)$: (x and $f(x)$ are neighbours in the graph)
2. $x \preceq f(x)$: ($f(x)$ is to the future of x with respect to the partial order)
3. for all $y \sim f(x)$, we have $x \preceq y$: (any other neighbours of $f(x)$ are all to the future of x)

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- Condition 3 guarantees **no loops**: by measuring x before $f(x)$ we will never have some y that $f(f(x)) = y$ and $y \preceq x$

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- **Example**: 2-dim lattice. $f(x) \Rightarrow$ same row, next column

Graph States as Stabiliser States

Graph state $|G\rangle$ is defined as:

$$|G\rangle = \prod_{(a,b)\in E} \wedge Z^{(a,b)} |+\rangle^{\otimes V}$$

An operator A **stabilises** a state if ($\lambda = +1$ eigenspace)

$$A|\psi\rangle = |\psi\rangle$$

The state $|G\rangle$ is a **stabiliser state** with generators:

$$K_i := X_i \left(\prod_{j\in N_G(i)} Z_j \right)$$

For each vertex $i \in V$ there is a stabiliser that has X at that vertex and Z to all its neighbours $N_G(i)$ in the graph.

Graph States as Stabiliser States

Operators K_i stabilise $|G\rangle$:

$$\begin{aligned} K_i |G\rangle &= X_i \left(\prod_{j \in N_G(i)} Z_j \right) \prod_{(a,b)\in E} \wedge Z^{(a,b)} |+\rangle^{\otimes V} \\ &= X_i \prod_{(a,b)\in E} \wedge Z^{(a,b)} |+\rangle^{\otimes V \setminus N_G(i)} |-\rangle^{\otimes j \in N_G(i)} \end{aligned}$$

where we used that Z commutes with $\wedge Z$ and that $Z|+\rangle = |-\rangle$.

Graph States as Stabiliser States

We know that $X_i \wedge Z^{(i,j)} = \wedge Z^{(i,j)} X_i Z_j$ and we get:

$$K_i |G\rangle = \prod_{(a,b)\in E} \wedge Z^{(a,b)} \left(X_i \prod_{j\in N_G(i)} Z_j \right) |+\rangle^{\otimes V \setminus N_G(i)} |-\rangle^{\otimes j \in N_G(i)}$$

since X_i acts as above if i belongs to that edge while it commutes with all the other $\wedge Z$ that do not involve qubit i. However this changes back the states since $Z|-\rangle = |+\rangle$, and $X|+\rangle = |+\rangle$ results to

$$K_i |G\rangle = \prod_{(a,b)\in E} \wedge Z^{(a,b)} |+\rangle^{\otimes V}, \ \forall\ i \in V = |G\rangle$$

- It can be shown that this set of generators uniquely determines the graph state $|G\rangle$.

Part II

How to cancel the "corrections" due to randomness

How to apply an operator by acting on different qubits

Given a graph state $|G\rangle$, we can apply X, Z operators at qubit i by acting on qubits other than i.

We will use: $K_i |G\rangle = |G\rangle$

1. To apply X_i:

$$X_i |G\rangle = X_i K_i |G\rangle = \prod_{j \in N_G(i)} Z_j |G\rangle$$

   where $N_G(i)$ are the neighbours of i in the graph

2. To apply Z_i:

$$Z_i |G\rangle = Z_i K_{f(i)} |G\rangle = X_{f(i)} \prod_{j \in N_G(f(i)) \setminus i} Z_j |G\rangle$$

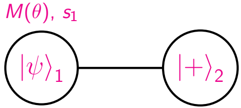
   where $f(i)$ is the flow of i and $N_G(f(i)) \setminus i$ are all the neighbours of $f(i)$ in the graph apart from i

How to apply an operator by acting on different qubits

Can cancel a correction **after** measuring that qubit, provided **all** qubits j involved are still not measured!

- This **cannot** be done for the X operator (qubits both to the past and future according to the definition of flow).
  Need to adapt the angles
- This **can** be done for the Z operator because of properties of the flow! (conditions 2 and 3)

Rules for adapting measurement angles



Measuring at θ qubit 1

Recall (previous lecture) the state before measurement is:

$$
\begin{aligned}
|\chi\rangle_{12} &= \wedge Z_{12} \left(|\psi\rangle_1 \otimes |+\rangle_2 \right) = a\,|0+\rangle_{12} + b\,|1-\rangle_{12} \\
&= |+_\theta\rangle_1\, X^{s_1=0} J(-\theta)_2\, |\psi\rangle_2 + |-_\theta\rangle_1\, X_2^{s_1=1} J(-\theta)_2\, |\psi\rangle_2
\end{aligned}
$$

Rules for adapting measurement angles

If we could have started with $Z_1^{s_1}|\psi\rangle_1$ state instead of $|\psi\rangle_1$:

$$\begin{aligned}|\chi\rangle_{12} &= |+_\theta\rangle_1 X^{s_1} J(-\theta)_2 Z_2^{s_1}|\psi\rangle_2 + |-_\theta\rangle_1 X_2^{s_1} J(-\theta)_2 Z_2^{s_1}|\psi\rangle_2 \\ &= |+_\theta\rangle_1 J(-\theta)_2|\psi\rangle_2 + |-_\theta\rangle_1 J(-\theta)_2|\psi\rangle_2\end{aligned}$$

using that $J(-\theta)Z^{s_1} = X^{s_1}J(-\theta)$ and $X^{s_1}X^{s_1} = \mathbf{I}$. Now, there is **no random correction** and any outcome of the measurement leads to the desired gate.

- Getting the "wrong" outcome $s_i = 1$ is as if a Z-correction on the initial state was applied, and could cancel it by applying another Z on that qubit.
- However, to do this we **need to know** s_1 which is the outcome of measuring qubit 1, and this (clearly) happens **after** the preparation of qubit 1.

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- If the "correction" is on output qubit: see next part
- If the "correction is on non-output qubits, instead of acting on them, we can modify the measurement angles:

$$M_i^{\phi_i} X = M_i^{-\phi_i} \; ; \; M_i^{\phi_i} Z = M_i^{\phi_i + \pi}$$

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This holds since:

$X \left|+_\phi\right\rangle = e^{i\phi} \left|+_{(-\phi)}\right\rangle$ and $Z \left|+_\phi\right\rangle = \left|-_\phi\right\rangle = \left|+_{(\phi+\pi)}\right\rangle$

E.g. $\left|+_\phi\right\rangle \left\langle +_\phi\right| X e^{i\phi} = \left|+_\phi\right\rangle \left\langle +_{-\phi}\right|$ and $\left|+_\phi\right\rangle \left\langle +_\phi\right| Z = \left|+_\phi\right\rangle \left\langle +_{\phi+\pi}\right|$

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     The set of vertices that induce X corrections to qubit i is denoted as $S_x(i) = \{f^{-1}(i)\}$.
  2. A Z-correction from all qubits $j \neq i$ that their flow $f(j)$ is neighbour to i.

     I.e. take each k of the neighbours of i and find $j := f^{-1}(k)$

     The set of vertices that induce Z corrections to qubit i is denoted as $S_z(i) = \{j \neq i \ : \ i \in N_G(f(j))\}$.

Rules for adapting measurement angles

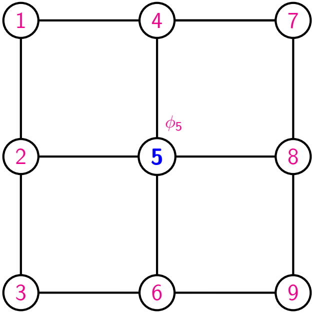
This sum leads to the corrected angle ϕ_i':

$$\phi_i' = (-1)^{\sum_{j \in S_x(i)} s_j} \phi_i + \pi \left(\sum_{j \in S_z(i)} s_j \right)$$

$$\phi_i' = (-1)^{s_{f^{-1}(i)}} \phi_i + \pi \left(\sum_{j : i \in N_G(f(j)) | j \neq i} s_j \right)$$

An Example, Output Corrections and MBQC recap

Example



Adapting measurement angles

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- Consider qubit 5:
  - $f^{-1}(5) = 2$
  - $N_G(5) = \{2, 4, 6, 8\}$
  - X-correction from s_2, i.e. $S_x(5) = \{2\}$
  - Z-corrections from: $\{s_1, s_3\}$, i.e. $S_z(5) = \{1, 3\}$. Since we look for $f^{-1}(\cdot)$ for each of the neighbours of 5. Qubit 2 has no past, qubit 8 has our qubit to its past, which leaves only the past of qubit 4 and qubit 6.

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  We then obtain the corrected measurement angle:

$$\phi_5' = (-1)^{s_2}\phi_5 + \pi(s_1 + s_3)$$

Note: Depends on outcomes measured **before** qubit 5.

Corrections on Output Qubits

- **Note:** Some algorithms have no output qubits (end with measurement)
- For the rest algorithms, output qubits are treated differently
- Output qubits cannot be corrected with adapting the measurement angle
- There is an operation that needs to be applied. Conditional on previous outcomes, one applies a X operation, a Z operation, both or none.

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- There is an operation that needs to be applied. Conditional on previous outcomes, one applies a X operation, a Z operation, both or none.
- **Correction** for qubit i: $X_i^{s_{X,i}} Z_i^{s_{Z,i}}$

  $s_{X,i}$ is affected by qubits in $S_x(i)$ that is defined as for the measured qubits

  $s_{Z,i}$ is affected by qubits in $S_z(i)$ that is defined as for the measured qubits

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- The coefficient of the X correction of output qubit i, depends only on $f^{-1}(i)$
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- **NOTE.** Measured Vs Output qubits

  Same sets of $S_x(i)$ and $S_z(i)$ qubits that influence them

  Measured qubits modify their measurement angle to ϕ'_i

  Output qubits need to apply the extra gates Z^{s_Z} and X^{s_X}

MBQC Summary

- Start with a universal **graph state**
- The computation is performed by **measuring** one-by-one the qubits using **single-qubit** bases: $\{|+_\theta\rangle, |-_\theta\rangle\}$ (or $\{|0\rangle, |1\rangle\}$).
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- Exact measurement basis depends on previous outcomes.
- The **order of measurements** is determined by the **flow** $f(i)$ along with partial order $\preceq$.
- Single qubit gates performed using $J(\theta)$-gate and along with $\wedge Z$ is **universal**. Random corrections need to be cancelled.
- A unitary U is implemented by a set of **default** angles ϕ_i if all measurement had outcomes $s_i = 0$.
- The actual **"corrected" basis** that a qubit i is measured is modified, using flow & stabiliser properties:

$$\phi_i' = (-1)^{s_{f^{-1}(i)}}\phi_i + \pi\left(\sum_{j:i\in N_G(f(j))} s_j\right)$$

Further Reading on MBQC

1. One-way Quantum Computation - a tutorial introduction, D. Browne and H. Briegel, arxiv:quant-ph/0603226
2. An introduction to measurement based quantum computation, R. Jozsa, arxiv:quant-ph/0508124
3. Quantum computing with photons: introduction to the circuit model, the one-way quantum computer, and the fundamental principles of photonic experiments, S. Barz, Journal of Physics B: Atomic, Molecular and Optical Physics, Vol 48, Num. 8 (2015).
4. Chapter 7, Semantic Techniques in Quantum Computation – Editors Simon Gay and Ian Mackie