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Tutorial 0

Problem 1: Complex Numbers

Consider the two complex numbers $v_1 = 1 + i$ and $v_2 = 1 - 2i$ where $i^2 = -1$.

a. Calculate the complex numbers $z_1 = v_1 + v_2$ and $z_2 = v_1 - v_2^*$ where z^* denotes the complex conjugate of the complex number z.

Solution: $z_1 = v_1 + v_2 = (1 + i) + (1 - 2i) = 2 - i$. In order to calculate z_2, we first have to conjugate the number v_2. Recall that for a complex number $w = a + bi$, its complex conjugate is $w^* = a - bi$. It's easy then to see that $v_2^* = 1 + 2i$ and thus $z_2 = -i$

b. Let $w = 1 - i$. Calculate wz_1 and $(z_2 w)^*$.

Solution: For the first multiplication we have:

$$wz_1 = (1 - i)(2 - i) = 2 - i - 2i - 1 = 1 - 3i$$

since $i^2 = -1$. For the second expression, we should first do the multiplication and then calculate the conjugate of the product. So

$$z_2 w = -i(1 - i) = -i - 1 = -1 - i$$

and then if we conjugate:

$$(z_2 w)^* = -1 + i$$

c. Calculate the norm of v_1 and v_2.

Solution: The *norm* of complex number $w = a + bi$ is defined as

$$|w| = \sqrt{a^2 + b^2}$$

In our case, for v_1:

$$|v_1| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

and for v_2:

$$|v_2| = \sqrt{1^2 + (-2)^2} = \sqrt{5}$$

Problem 2: Inner-product and orthonormal bases

a. Consider the quantum states $|R\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}$, $|L\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}$,

1. Write $\langle R|$ and $\langle L|$ in vector notation.
2. Prove that both $|R\rangle$ and $|L\rangle$ are normalized, i.e. $\sqrt{\langle R|R\rangle} = \sqrt{\langle L|L\rangle} = 1$
3. Are $|R\rangle$ and $|L\rangle$ orthogonal?

4. Show that $|R\rangle$ and $|L\rangle$ satisfy all the conditions of an orthonormal basis of $\mathcal{H} = \mathbb{C}^2$.

Solution:

Let a vector $|\psi\rangle$ in the Dirac "ket" notation. If $|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$ then, the conjugate transpose vector, denoted $\langle\psi|$ and called a "bra" is defined as $\langle\psi| = (|\psi\rangle^T)^* = |\psi\rangle^\dagger = \begin{pmatrix} a^* & b^* \end{pmatrix}$ Thus,

$$\langle R| = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 - i \end{pmatrix}$$

and

$$\langle L| = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \end{pmatrix}$$

We will prove that both $|R\rangle$, $|L\rangle$ are normalised.

$$\langle R|R\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2}(1+1) = 1$$

and so:

$$\sqrt{\langle R|R\rangle} = 1$$

Same for $|L\rangle$:

$$\langle L|L\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{1}{2}(1+1) = 1$$

and so:

$$\sqrt{\langle L|L\rangle} = 1$$

Two vectors $|R\rangle$, $|L\rangle$ are *orthogonal* if their inner product is 0, i.e. $\langle R|L\rangle = 0$. We have,

$$\langle R|L\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{1}{2}(1-1) = 0$$

So $|R\rangle$ and $|L\rangle$ are *orthogonal*.

Finally, for the last question, in order for $|R\rangle$ and $|L\rangle$ to satisfy all the conditions of an orthonormal basis, they must satisfy:

- Be orthogonal, which is true as we proved before.
- Be normalized to one, which we proved to be.
- The number of basis elements must be the same with the dimension of the vector space which is true as well.

Problem 3: Matrices and operators.

a.

1. One of the most important linear operators in quantum computing is the *Hadamard operator* defined as:

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Find what is the action of the operator on the vector $|v\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$.

Solution. We want to calculate $H|v\rangle$. We have:

$$H|v\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+i \\ 1-i \end{pmatrix}$$

2. Consider two of the Pauli matrices:

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Calculate XZ and ZX. Compare the two calculations.

Solution: We start by computing XZ. We have:

$$XZ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

We continue by computing ZX. We have:

$$ZX = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

If we observe the two multiplications we see that $ZX = -XZ$. This is a well-known property of the Pauli matrices as all of them *anticommute*. For our case this translates to $\{X, Z\} = XZ + ZX = 0$.

b.

1. Show that for finite-size matrices $(A^\dagger)^\dagger = A$ always holds.

Solution. Since $A^\dagger_{ij} = A^*_{ji}$ then $(A^\dagger_{ij})^\dagger = (A^*_{ji})^\dagger = (A^*_{ij})^* = A_{ij}$ and thus:

$$(A^\dagger)^\dagger = A \text{ for every operator } A$$

2. Prove that two general matrices A and B we have $(AB)^\dagger = B^\dagger A^\dagger$.

   Solution: The definition of an adjoint operator M is:

$$(|v\rangle, M|w\rangle) = (M^\dagger|v\rangle, |w\rangle)$$

   We can now write

$$(|v\rangle, AB|u\rangle) = (|v\rangle, A(B|u\rangle)),$$

   where setting $|w\rangle = B|u\rangle$ and $M = A$ in the definition of adjoint operator above, allows us to write

$$(|v\rangle, A(B|u\rangle)) = (A^\dagger|v\rangle, B|u\rangle).$$

   Using again the definition of the adjoint operator, now with B, we obtain

$$(A^\dagger|v\rangle, B|u\rangle) = (B^\dagger A^\dagger|v\rangle, |u\rangle)$$

   and

$$\begin{aligned}(|v\rangle, AB|u\rangle) &= ((AB)^\dagger|v\rangle, |u\rangle)\\ \implies (AB)^\dagger &= B^\dagger A^\dagger\end{aligned}$$

3. Prove that the Hadamard operator defined above is a self-adjoint operator.

   Solution. As we already mentioned, the elements of the adjoint Hadamard operator $H^\dagger$ are related to those of the Hadamard operator H as $H^\dagger_{ij} = H^*_{ji}$. It is clear then that these two matrices are identical and as such the Hadamard operator is a *self-adjoint operator*.

c. Compute the eigenvalues and eigenvectors of X and Z.

Solution. We will work with the matrix X. The eigenvectors $|v\rangle$ of the matrix X are such that when X acts on the vectors $|v\rangle$ they are only scaled by a factor λ (which is called the eigenvalue of the matrix), i.e. $X|v\rangle = \lambda|v\rangle$

The eigenvalues λ of the matrix X must satisfy:

$$\begin{aligned}\det(X - \lambda I) = 0 &\implies \left|\begin{pmatrix}-\lambda & 1\\ 1 & -\lambda\end{pmatrix}\right| = 0\\ &\implies \lambda^2 - 1 = 0 \implies \lambda = \pm 1\end{aligned}$$

Thus, we found that the eigenvalues of X are ± 1. In order to find the eigenvectors, we replace the eigenvalues in the equation $X|v\rangle = \lambda|v\rangle$. Let's also write the vectors $|v\rangle$ as $|v\rangle = \begin{pmatrix}a\\ b\end{pmatrix}$.

For $\lambda = 1$ we have:

$$\begin{aligned}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} &= \begin{pmatrix}a\\ b\end{pmatrix}\\ \implies \begin{pmatrix}b\\ a\end{pmatrix} &= \begin{pmatrix}a\\ b\end{pmatrix}\end{aligned}$$

We can then conclude that the eigenvector corresponding to $\lambda = 1$ eigenvalue is $|v\rangle = \begin{pmatrix} a \\ a \end{pmatrix}$. If we impose the condition that the vector is normalized $|| \, |v\rangle \, || = 1$ then we get $a = \frac{1}{\sqrt{2}}$. So the eigenvector becomes $|v\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

By working in the same manner for the second eigenvalue ($\lambda = -1$) it is easy to see that the second eigenvector is $|u\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. In the quantum computing literature you will find that these two vectors are usually denoted as $|+\rangle$ and $|-\rangle$.

It is trivial to see that for Z the eigenvectors are the states of the computational basis $|0\rangle$ and $|1\rangle$ with eigenvalues 1 and -1 respectively.

Optional: More complex numbers

a. Use the *Euler equation*, i.e. $e^{i\theta} = \cos\theta + i\sin\theta$, to calculate $e^{i\pi}$ and $e^{2i\pi/4}$.

Solution: For the first case, $\theta = \pi$ and thus:

$$e^{i\pi} = \cos\pi + i\sin\pi = -1 + i0 = -1$$

For the second case, $\theta = 2\pi/4 = \pi/2$

$$e^{i2\pi/4} = \cos(\pi/2) + i\sin(\pi/2) = 0 + i = i$$

b. Let $z = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$. First calculate $|z|$ and then use the Euler equation to obtain ϕ so that $z = |z|e^{i\phi}$.

Solution: As mentioned in question c. the norm of a complex number $w = a + bi$ is defined as:

$$|w| = \sqrt{a^2 + b^2}$$

For $z = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$:

$$|z| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$$

So we need to find the angle θ so that $z = |z|e^{i\phi} = e^{i\phi}$ (since $|z| = 1$). Using the Euler equation:

$$\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i = \cos\phi + i\sin\phi$$
$$\left(\frac{1}{\sqrt{2}} - \cos\phi\right) - \left(\frac{1}{\sqrt{2}} + \sin\phi\right)i = 0$$

For a complex number $w = a + bi$ to be equal to zero, it must have both its imaginary and real part equal to zero. First, for the real part:

$$\cos\phi = \frac{1}{\sqrt{2}}$$

and for the imaginary part:

$$\sin\phi = -\frac{1}{\sqrt{2}}$$

Thus $\phi = 7\pi/4$ and $z = e^{i7\pi/4}$