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Tutorial 1

Problem 1: Quantum Operations

One of the most important linear operators in quantum computing is the *Hadamard operator* defined as:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

a. Prove that H is unitary, i.e. that it satisfies $HH^\dagger = H^\dagger H = I$.

Solution: In order to prove that a matrix U is unitary, it must satisfies that $UU^\dagger = U^\dagger U = I$.

First we have to calculate the adjoint of the Hadamard operator. Recall that the matrix elements of the adjoint operator are related to that of the operator as $H^\dagger_{ij} = H^*_{ji}$. Thus:

$$H^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

So we can see that $H^\dagger = H$. In order for H to be unitary then the following must hold:

$$H^\dagger H = HH^\dagger = I$$

We have:

$$HH^\dagger = H^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = I$$

and thus $H^\dagger H = HH^\dagger = I$

b. Prove that H is its own inverse by showing $H^2 = I$ where I is the identity operator.

Solution: This is a corollary of the previous result.

c. Calculate the action of the operator on the vectors:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, |+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, |-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Solution: We will show what is the action of the Hadamard on the computational basis vector $|0\rangle$ and on the vector $|+\rangle$.

$$H|0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = |+\rangle$$

We can see how H acts on the $|+\rangle$ by doing the matrix multiplication but we can think of a more "clever" way. We proved on question b. that $H^2 = I$. So,

$$H|+\rangle = H(H|0\rangle) = H^2|0\rangle = |0\rangle$$

You can work in the same way with the other two examples and prove that $H|1\rangle = |-\rangle$ and that $H|-\rangle = |1\rangle$.

Extra information: $H^\dagger = H$ makes Hadamard a *Hermitian operator* and so $H^\dagger H = HH^\dagger$. The operators that satsify $AA^\dagger = A^\dagger A$ are called *normal operators.*

Problem 2: Pauli matrices

Consider the four Pauli matrices:

$$I, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

a. Prove that for each Pauli matrix σ_i we have $\sigma_i^2 = I$ and $\sigma_i^\dagger = \sigma_i$.

Solution: We'll only make the proof for the Y Pauli matrix but you should do the exact calculations on the rest. We have:

$$Y^2 = YY = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

We will also prove that Y satisfies $Y^\dagger = Y$ (i.e., is Hermitian):

$$Y^\dagger = \left[\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}^T \right]^* = \left[\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \right]^* = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = Y$$

b. Show that the Pauli matrices are unitary matrices.

Solution: We proved that for all Pauli matrices $\sigma_i^\dagger = \sigma_i$ and that $\sigma_i^2 = I$. Clearly then, $\sigma_i^2 = \sigma_i \sigma_i = \sigma_i^\dagger \sigma_i = \sigma_i \sigma_i^\dagger = I$.

c. Show that $Y = iXZ$.

Solution: We have:

$$iXZ = i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = Y$$

d. Show that $HXH = Z$ and $HZH = X$.

Solution: First, we will prove that $HXH = Z$. We have:

$$\begin{aligned} HXH &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} = Z \end{aligned}$$

We can work in the exact same way for $HZH = X$ or we can prove it in a different way. We proved that $HXH = Z$. We do a left and right multiplication with H and so we have $HHXHH = HZH$. But $H^2 = I$ and so $X = HZH$.

Problem 3: Measurement

Consider the two quantum states $|L\rangle$ and $|R\rangle$ of Problem 1 (these two quantum states are the eigenvalues of Pauli Y operator):

$$|R\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle)$$
$$|L\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i\,|1\rangle)$$

a. Consider the general quantum state:

$$|\psi\rangle = \psi_0\,|0\rangle + \psi_1\,|1\rangle$$

What are the probabilities of outcome $|R\rangle$ and $|L\rangle$ if we measure $|\psi\rangle$.

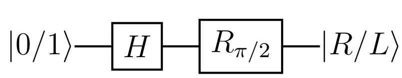
Solution. We start with the probability of measuring the outcome $|R\rangle$. If we measure the state $|\psi\rangle$, the probability of measuring $|R\rangle$ is given by:

$$\begin{aligned} Pr[R] = |\langle R|\psi\rangle|^2 &= \left|\frac{1}{\sqrt{2}}(\langle 0| - i\,\langle 1|)(\psi_0\,|0\rangle + \psi_1\,|1\rangle)\right|^2 \\ &= \frac{1}{2}\,|\psi_0 - i\psi_1|^2 \end{aligned}$$

where we used $\langle 0|1\rangle = 0$, since the vectors are orthogonal. Similarly, for the other probability we have:

$$\begin{aligned} Pr[L] = |\langle L|\psi\rangle|^2 &= \left|\frac{1}{\sqrt{2}}(\langle 0| + i\,\langle 1|)(\psi_0\,|0\rangle + \psi_1\,|1\rangle)\right|^2 \\ &= \frac{1}{2}\,|\psi_0 + i\psi_1|^2 \end{aligned}$$

b. Show that the states $|L\rangle$ and $|R\rangle$ can be generated from $|0\rangle$ and $|1\rangle$ using the following circuit:



where

$$R_\theta = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}.$$

Solution:

$$R_{\pi/2}H\,|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = |R\rangle$$

$$R_{\pi/2}H|1\rangle = \begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix} = |L\rangle$$

c. What circuit will allow to implement a measurement on the $|L\rangle$ and $|R\rangle$ basis if our hardware only allows for measurement in the computational basis but Hadamard gates and R_θ.

Solution: In 1a) we saw that H is its own inverse and it can be verified that $R_{-\pi/4}$ is inverse of $R_{\pi/4}$ gate, i.e. $R_{-\pi/4}R_{\pi/4} = R_{\pi/4}R_{-\pi/4} = I$. Hence the circuit

$$|R/L\rangle — \boxed{R_{-\pi/2}} — \boxed{H} — |0/1\rangle$$

can be used to map $|R\rangle$ and $|L\rangle$ into a computational basis where they can be distinguished by the measurement.

Problem 4: Outer-product and projectors

a. Show that the following matrices can be written as the out-product of the the $|+\rangle$ and $|-\rangle$ states:

$$|+\rangle\langle+| = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}; |-\rangle\langle-| = \frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}.$$

Solution: We express the states in the matrix form: $|+\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}$, $\langle+| = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}$ and $|-\rangle = \frac{|0\rangle-|1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}$, $\langle-| = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\end{pmatrix}$ and the result follows by direct computation.

b. Show that $P_+ = |+\rangle\langle+|$ and $P_- = |-\rangle\langle-|$ are projectors by verifying the condition $P_i^2 = P_i$ and they project on orthogonal basis as $P_+P_- = 0$.

Solution: This can be checked either algebraically $P_+^2 = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$ and similarly for P_-; or by using the braket notation which simplifies the proof even more:
$P_+^2 = (|+\rangle\langle+|)(|+\rangle\langle+|) = |+\rangle\langle+|+\rangle\langle+| = |+\rangle 1\langle+| = |+\rangle\langle+| = P_+$
$P_-^2 = (|-\rangle\langle-|)(|-\rangle\langle-|) = |-\rangle\langle-|-\rangle\langle-| = |-\rangle 1\langle-| = |-\rangle\langle-| = P_-$. Orthogonality can also be checked the two different ways, the braket one being: $P_+P_- = (|+\rangle\langle+|)(|-\rangle\langle-|) = |+\rangle\langle+|-\rangle\langle+| = |+\rangle 0\langle-| = 0$

c. Check the completeness relation for measurement on the $\{|+\rangle, |-\rangle\}$ basis.
Solution: Since P_+ and P_- are orthogonal projectors onto $\{|+\rangle, |-\rangle\}$ basis, it remains to check if they satisfy the completeness relation to be valid projective measurement operators.

I.e., it should hold that $P_+ + P_- = |+\rangle\langle+| + |-\rangle\langle-| = I$ what can be checked by direct calculation:

$$|+\rangle\langle+| + |-\rangle\langle-| = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

d. Compute $P(+) = ||P_+|\psi\rangle||^2$ and $P(-) = ||P_-|\psi\rangle||^2$ for an arbitrary state $|\psi\rangle = \psi_0|0\rangle + \psi_1|1\rangle$ and show that $P(+) + P(-) = 1$, as expected.

Solution:

Using the definition of the norm $||\,|\psi\rangle\,|| = \sqrt{\langle\psi|\psi\rangle}$ and $\alpha^*\alpha = |\alpha|^2$ for any complex number α, we can show:

$$\begin{aligned}
P(+) &= ||P_+\,|\psi\rangle\,||^2 = ||\,|+\rangle\langle+|\,(\psi_0\,|0\rangle + \psi_1\,|1\rangle)||^2 \\
&= ||\psi_0\,|+\rangle\langle+|0\rangle + \psi_1\,|+\rangle\langle+|1\rangle\,||^2 \\
&= ||\frac{1}{\sqrt{2}}\psi_0\,|+\rangle + \frac{1}{\sqrt{2}}\psi_1\,|+\rangle\,||^2 \\
&= ||\frac{\psi_0 + \psi_1}{\sqrt{2}}\,|+\rangle\,||^2 \\
&= \frac{|\psi_0 + \psi_1|^2}{2}||\,|+\rangle\,||^2 \\
&= \frac{|\psi_0 + \psi_1|^2}{2}
\end{aligned}$$

$$\begin{aligned}
P(-) &= ||P_-\,|\psi\rangle\,||^2 = ||\,|-\rangle\langle-|\,(\psi_0\,|0\rangle + \psi_1\,|1\rangle)||^2 \\
&= ||\psi_0\,|-\rangle\langle-|0\rangle + \psi_1\,|-\rangle\langle-|1\rangle\,||^2 \\
&= ||\frac{1}{\sqrt{2}}\psi_0\,|-\rangle - \frac{1}{\sqrt{2}}\psi_1\,|-\rangle\,||^2 \\
&= ||\frac{\psi_0 - \psi_1}{2}\,|-\rangle\,||^2 \\
&= \frac{|\psi_0 - \psi_1|^2}{2}||\,|-\rangle\,||^2 \\
&= \frac{|\psi_0 - \psi_1|^2}{2}
\end{aligned}$$

Therefore

$$\begin{aligned}
P(+) + P(-) = \frac{|\psi_0 + \psi_1|^2}{2} + \frac{|\psi_0 - \psi_1|^2}{2} &= \frac{1}{2}((\psi_0 + \psi_1)(\psi_0 + \psi_1)^* + (\psi_0 - \psi_1)(\psi_0 - \psi_1)^* \\
&= \frac{1}{2}((\psi_0 + \psi_1)(\psi_0^* + \psi_1^*) + (\psi_0 - \psi_1)(\psi_0^* - \psi_1^*) \\
&= \frac{1}{2}(|\psi_0|^2 + \psi_0\psi_1^* + \psi_0^*\psi_1 + |\psi_1|^2 + |\psi_0|^2 - \psi_0\psi_1^* - \psi_0^*\psi_1 + |\psi_1|^2) \\
&= \frac{1}{2}(2|\psi_0|^2 + 2|\psi_1|^2) \\
&= |\psi_0|^2 + |\psi_1|^2 \\
&= 1
\end{aligned}$$