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Tutorial 1

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Problem 1: Quantum Operations

The *Hadamard* gate play a very prominent role in quantum computation:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

a. Prove that H is unitary, i.e. that it satisfies $UU^\dagger = U^\dagger U = I$.

b. Prove that H is it own inverse by showing $H^2 = I$ where I is the identity operator.

c. Calculate the action of the operator on the vectors:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, |+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, |-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Problem 2: Pauli matrices

Consider the four Pauli matrices:

$$I, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

a. Prove that for each Pauli matrix σ_i we have $\sigma_i^2 = I$ and $\sigma_i^\dagger = \sigma_i$.

b. Show that the Pauli matrices are unitary matrices.

c. Show that $Y = iXZ$.

d. Show that $HXH = Z$ and $HZH = X$.

Problem 3: Measurement

Consider the two quantum states $|L\rangle$ and $|R\rangle$ (eigenvalues of Pauli Y operator):

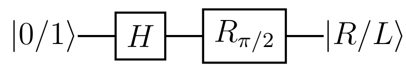
$$|R\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle)$$
$$|L\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i\,|1\rangle)$$

a. Consider the general quantum state:

$$|\psi\rangle = \psi_0\,|0\rangle + \psi_1\,|1\rangle$$

What are the probabilities of outcome $|R\rangle$ and $|L\rangle$ if we measure $|\psi\rangle$.

b. Show that the states $|L\rangle$ and $|R\rangle$ can be generated from $|0\rangle$ and $|1\rangle$ using the following circuit:



where

$$R_\theta = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}.$$

c. What circuit will allow to implement a measurement on the $|L\rangle$ and $|R\rangle$ basis if our hardware only allows for measurement in the computational basis but Hadamard gates and R_θ.

Problem 4: Outer-product and projectors

a. Show that the following matrices can be written as the out-product of the the $|+\rangle$ and $|-\rangle$ states:

$$|+\rangle\langle+| = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}; |-\rangle\langle-| = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

b. Show that $P_+ = |+\rangle\langle+|$ and $P_- = |-\rangle\langle-|$ are projectors by verifying the condition $P_i^2 = P_i$ and they project on orthogonal s as $P_+P_- = 0$.

c. Check the completeness relation for the measurement on the $\{|+\rangle, |-\rangle\}$ basis.

d. Compute $P(+) = ||P_+|\psi\rangle||^2$ and $P(-) = ||P_-|\psi\rangle||^2$ for an arbitrary state $|\psi\rangle = \psi_0|0\rangle + \psi_1|1\rangle$ satisfying the normalization condition and show that $P(+) + P(-) = 1$, as expected.