Raul Garcia-Patron
Petros Wallden
Milos Prokop

Tutorial 2

IQC 2022-23
October 5, 2023

Problem 1: Tensor Product

a. Consider the quantum state:

$$|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{\sqrt{1}}{2}|1\rangle$$

1. Calculate $|\psi\rangle^{\otimes 2}$, where $|\psi\rangle^{\otimes 2} \equiv |\psi\rangle \otimes |\psi\rangle$.
2. Calculate $|+\rangle \otimes |-\rangle \otimes |+\rangle$, where $|\pm\rangle = 1/\sqrt{2}(|0\rangle \pm |1\rangle)$.

b. Consider the four Pauli matrices:

$$I, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Calculate the tensor products:

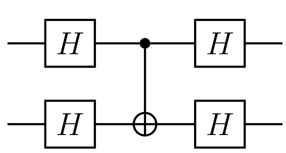
1. $Z \otimes X$
2. $X \otimes Y$
3. Compute the product of the two 4×4 matrices $(Z \otimes X) \times (X \otimes Y)$.
4. Use the identity $(A \otimes B)(C \otimes D) = (AC \otimes BD)$ to show that $(Z \otimes X) \times (X \otimes Y) = ZX \otimes XY$ and verify that the tensor product gives the same result as the multiplication above.

c. Consider two linear operators A, B.

1. Prove that if A, B are unitary operators then $(A \otimes B)$ is a unitary.
2. Prove that if A, B are projector operators then $(A \otimes B)$ is a projector.

Problem 2: Concatenation and composition of gates

Consider the quantum circuit which consists of two Hadamard gates H, followed by a CNOT and finally with two more Hadamard gates:



a. We have seen in the lectures that a CNOT can be written in terms of a linear combination of tensor products of projectors into the computation basis $P_{0/1}$, the identity matrix I and the Pauli matrix X. Using the rules of tensor product and linearity seen in the course to prove that this circuit is equivalent to reversed CNOT (control is on the lower qubit).

b. Calculate the output state $|\psi\rangle$ via application of the different quantum gates to the inputs:

1. $|\psi_1\rangle = a|0\rangle + b|1\rangle$ with $a, b \in \mathbb{C}$ is a general quantum state and $|\psi_2\rangle = |0\rangle$.
2. $|\psi_1\rangle = |0\rangle$ and $|\psi_2\rangle = a|0\rangle + b|1\rangle$ with $a, b \in \mathbb{C}$.

b. Compute the rules of concatenation and composition of the quantum gates the resulting quantum circuit unitary U. Can the above circuit be written as a well-known 2-qubit gate?

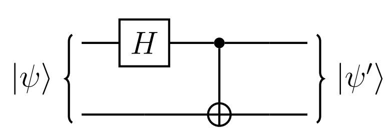
Problem 3: Measurement on Bell state basis

a. Consider the four Bell quantum states:

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), |\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$
$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), |\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

1. Verify that the Bell states form an orthonormal basis of the Hilbert system that describes the composite system.
2. Prove the *completeness relation* $\sum_{i=1}^{4} |u_i\rangle \langle u_i| = I_4$ where $|u_i\rangle$ are the set of four Bell states.

b. Consider the quantum circuit:

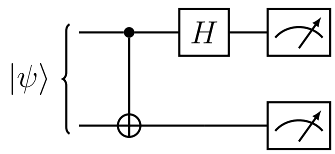


Calculate the output state when:

- $|\psi\rangle = |0\rangle|0\rangle$
- $|\psi\rangle = |0\rangle|1\rangle$
- $|\psi\rangle = |1\rangle|0\rangle$

- $|\psi\rangle = |1\rangle|1\rangle$

c. Consider the quantum circuit ending in a joint computational measurement of both qubits, leading to four possible outcomes $00, 01, 10$, and 11:



1. If we have as input to the circuit the Bell state $|\Psi^+\rangle$, what is the probability of the 4 possible outcomes?
2. And when we input $|\Psi^-\rangle$ and $|\Phi^\pm\rangle$ into the circuit?
3. What are the outcome probabilities resulting from the input state $|00\rangle$?
4. What will be the outcomes probabilities when we input any state of the computational basis of the two qubits, i.e., $|x_1\rangle \otimes |x_2\rangle$ where $x_i \in \{0, 1\}$?