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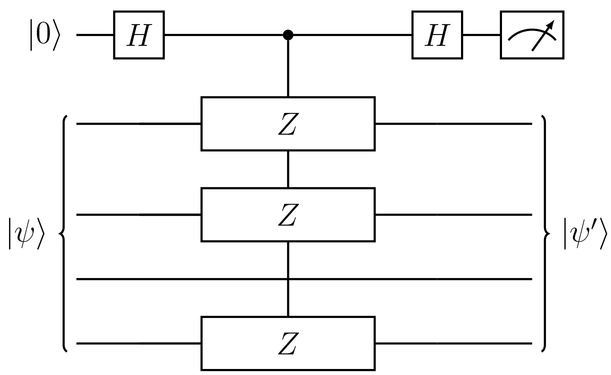
Tutorial 5

Problem 1: Three-Qubit Parity Check

We want to perform an even/odd parity check on qubits $1, 2, 4$. It's easy to see that the parity operator $P = Z \otimes Z \otimes I \otimes Z$ is both Hermitian and Unitary, so that it can both be regarded as an observable and a quantum gate. Suppose we wish to measure the observable P. That is, we desire to obtain a measurement result indicating one of the two eigenvalues, and leaving an updated state after the measurement that is projected to its corresponding eigenspace. We are going to show that the following circuit implements a measurement of P:



a. Derive the action of the three-qubit parity operator $P = Z \otimes Z \otimes I \otimes Z$ on the computational basis state $|x_1 x_2 x_3 x_4\rangle$. What are the eigenvalues of the operator P?

Solution: Recall that the state $|x_1 x_2 x_3 x_4\rangle$ corresponds to the tensor product:

$$|x_1 x_2 x_3 x_4\rangle \equiv |x_1\rangle \otimes |x_2\rangle \otimes |x_3\rangle \otimes |x_4\rangle$$

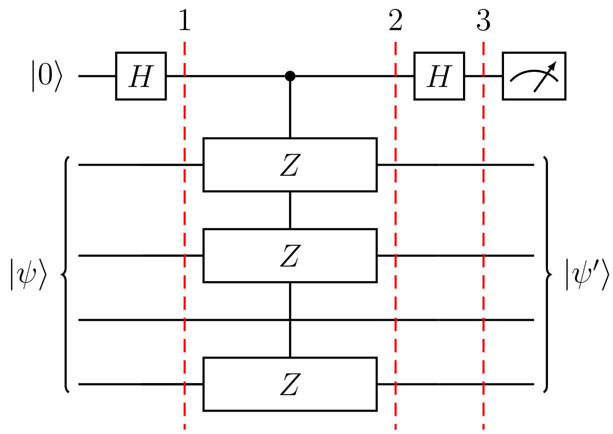
We can use the property:

$$\begin{gathered} P\,|x_1 x_2 x_3 x_4\rangle = (Z \otimes Z \otimes I \otimes Z)\,(|x_1\rangle \otimes |x_2\rangle \otimes |x_3\rangle \otimes |x_4\rangle) \\ = Z\,|x_1\rangle \otimes Z\,|x_2\rangle \otimes I\,|x_3\rangle \otimes Z\,|x_4\rangle = (-1)^{x_1}\,|x_1\rangle \otimes (-1)^{x_2}\,|x_2\rangle \otimes |x_3\rangle \otimes (-1)^{x_4}\,|x_4\rangle \\ \implies P\,|x_1 x_2 x_3 x_4\rangle = (-1)^{x_1+x_2+x_4}\,|x_1 x_2 x_3 x_4\rangle \end{gathered}$$

We can see that when P acts on a computational basis, it is scaled by a factor of -1 or $+1$ depending on the bits x_i. This means that the compuational basis states are the eigenvectors of P with eigenvalues ± 1.

b. Derive the global state right before the measurement of the upper-qubit when the input state reads $|0\rangle \otimes |\psi\rangle$, where $|\psi\rangle = \sum_{x \in \{0,1\}^4} \gamma_x |x\rangle$ is a four qubit arbitrary input state and x is a four bit string.

Solution: First of all, we are going to divide the quantum circuits into subsequent steps and calculate the composite state in each one of them.



The initial state of the composite system of 5 qubits is:

$$|\psi\rangle_0 = |0\rangle \otimes |\psi\rangle = \sum_{x\in\{0,1\}^4} \gamma_x |0\rangle |x\rangle$$

Step 1: On the first step, we act with the Hadamard operator on the first qubit and get:

$$(H \otimes I)|\psi\rangle_0 = \sum_{x\in\{0,1\}^4} \gamma_x H|0\rangle |x\rangle = \sum_{x\in\{0,1\}^4} \gamma_x \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|x\rangle$$
$$\sum_{x\in\{0,1\}^4} \frac{\gamma_x}{\sqrt{2}} |0\rangle |x\rangle + \sum_{x\in\{0,1\}^4} \frac{\gamma_x}{\sqrt{2}} |1\rangle |x\rangle$$

and so the state $|\psi\rangle_1$ at step 1 is:

$$|\psi\rangle_1 = \sum_{x\in\{0,1\}^4} \frac{\gamma_x}{\sqrt{2}} |0\rangle |x\rangle + \sum_{x\in\{0,1\}^4} \frac{\gamma_x}{\sqrt{2}} |1\rangle |x\rangle$$

Step 2: On the second step, we act with the controlled-P operator and get:

$$CP|\psi\rangle_1 = \sum_{x\in\{0,1\}^4} \frac{\gamma_x}{\sqrt{2}} |0\rangle |x\rangle + \sum_{x\in\{0,1\}^4} \frac{\gamma_x}{\sqrt{2}} |1\rangle P|x\rangle$$

Note that x is the bitstring $x_1x_2x_3x_4$. Thus, by using the answer of question (a.) we get that the state $|\psi\rangle_2$ at step 2 is:

$$\begin{aligned}|\psi\rangle_2 &= \sum_{x\in\{0,1\}^4} \frac{\gamma_x}{\sqrt{2}} |0\rangle |x\rangle + \sum_{x\in\{0,1\}^4} \frac{\gamma_x}{\sqrt{2}} |1\rangle (-1)^{x_1+x_2+x_4} |x\rangle \\ &= \sum_{x_1+x_2+x_4=even} \gamma_x |+\rangle |x\rangle + \sum_{x_1+x_2+x_4=odd} \gamma_x |-\rangle |x\rangle\end{aligned}$$

Step 3: On the third step, we act again with the Hadamard operator on the first qubit and get:

$$|\psi\rangle_3 = \sum_{x_1+x_2+x_4=even} \gamma_x |0\rangle |x\rangle + \sum_{x_1+x_2+x_4=odd} \gamma_x |1\rangle |x\rangle$$

$$|\psi\rangle_3 = |0\rangle \otimes \left(\sum_{x_1+x_2+x_4=even} \gamma_x |x\rangle \right) + |1\rangle \otimes \left(\sum_{x_1+x_2+x_4=odd} \gamma_x |x\rangle \right)$$

c. Using the rules of partial measurement, show that the measurement of the upper-qubit projects the state of the lower four qubits to its odd or even parity subspaces, depending of the outcome being 0 or 1.

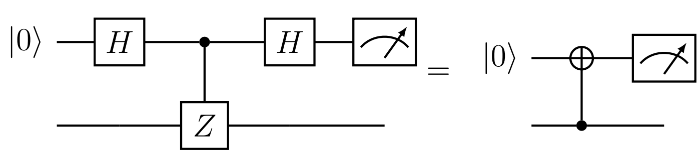
Solution: The partial measurement of the first qubit can be described as the linear operator $P_i \otimes I = |i\rangle \langle i| \otimes I$ with $i \in \{0, 1\}$. If we perform the measurement on the first qubit and find it in the $|0\rangle$ state, then the system after the measurement will be in the state:

$$|\psi\rangle = \frac{P_0 \otimes I |\psi\rangle_3}{||P_0 \otimes I |\psi\rangle_3||} = |0\rangle \otimes \frac{1}{(\sum_{x_1+x_2+x_4=even} |\gamma_x|^2)^{1/2}} \left(\sum_{x_1+x_2+x_4=even} \gamma_x |x\rangle \right)$$

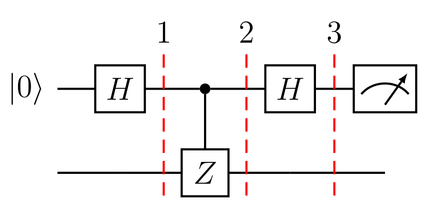
On the other hand, if we measure it to be in the state $|1\rangle$ then the state of the system after the measurement will be:

$$|\psi\rangle = \frac{P_1 \otimes I |\psi\rangle_3}{||P_1 \otimes I |\psi\rangle_3||} = |1\rangle \otimes \frac{1}{(\sum_{x_1+x_2+x_4=odd} |\gamma_x|^2)^{1/2}} \left(\sum_{x_1+x_2+x_4=odd} \gamma_x |x\rangle \right)$$

d. Prove that the two circuits below are equivalent:



Solution: Consider the second qubit to be in the general state $|\psi\rangle = a|0\rangle + b|1\rangle$. We split the first circuit into three parts.



The initial state of the composite system is:

$$|\psi\rangle_0 = a|00\rangle + b|01\rangle$$

Step 1:

$$|\psi\rangle_1 = (H \otimes I)\,|\psi\rangle_0 = \frac{a}{\sqrt{2}}(|00\rangle + |10\rangle) + \frac{b}{\sqrt{2}}(|01\rangle + |11\rangle)$$

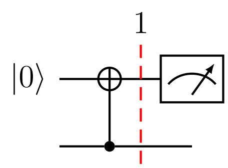
Step 2:

$$\begin{aligned}|\psi\rangle_2 = CZ\,|\psi\rangle_1 &= \frac{a}{\sqrt{2}}(|00\rangle + |10\rangle) + \frac{b}{\sqrt{2}}(|01\rangle - |11\rangle)\\ &= a\,|+\rangle\,|0\rangle + b\,|-\rangle\,|1\rangle\end{aligned}$$

Step 3:

$$|\psi\rangle_3 = (H \otimes I)\,|\psi\rangle_2 = a\,|0\rangle\,|0\rangle + b\,|1\rangle\,|1\rangle$$

We consider again the same input on the second circuit:



We can see that if we start with the same input:

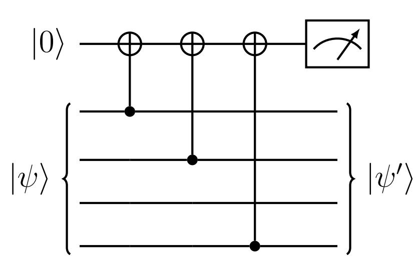
$$|\psi\rangle_0 = a\,|00\rangle + b\,|01\rangle$$

then after the action of the controlled-NOT with the control being the second qubit we have:

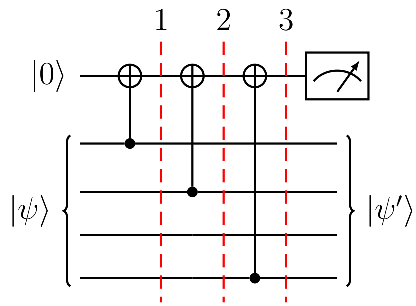
$$|\psi\rangle_1 = a\,|00\rangle + b\,|11\rangle$$

We can thus conclude that the two circuits are equivalent.

e. Prove that we can achieve the same result with the circuit:



Solution: In the same manner, we break the circuit into subsequent steps:



The initial state of the system is:

$$|\psi\rangle_0 = |0\rangle \otimes |\psi\rangle = \sum_{x\in\{0,1\}^4} \gamma_x |0\rangle |x\rangle$$

Step 1:

$$|\psi\rangle_1 = \sum_{x\in\{0,1\}^4} \gamma_x |0 \oplus x_1\rangle |x\rangle$$

Step 2:

$$|\psi\rangle_2 = \sum_{x\in\{0,1\}^4} \gamma_x |0 \oplus x_1 \oplus x_2\rangle |x\rangle$$

Step 3:

$$|\psi\rangle_3 = \sum_{x\in\{0,1\}^4} \gamma_x |0 \oplus x_1 \oplus x_2 \oplus x_4\rangle |x\rangle$$

$$\implies |\psi\rangle_3 = |0\rangle \otimes \left(\sum_{x_1+x_2+x_4=even} \gamma_x |x\rangle \right) + |1\rangle \otimes \left(\sum_{x_1+x_2+x_4=odd} \gamma_x |x\rangle \right)$$

We can see that in both cases the output state is the same. We can thus conclude that the two circuits are equivalent.

Problem 2: Syndrome Measurement

Consider the following quantum error correcting circuit:

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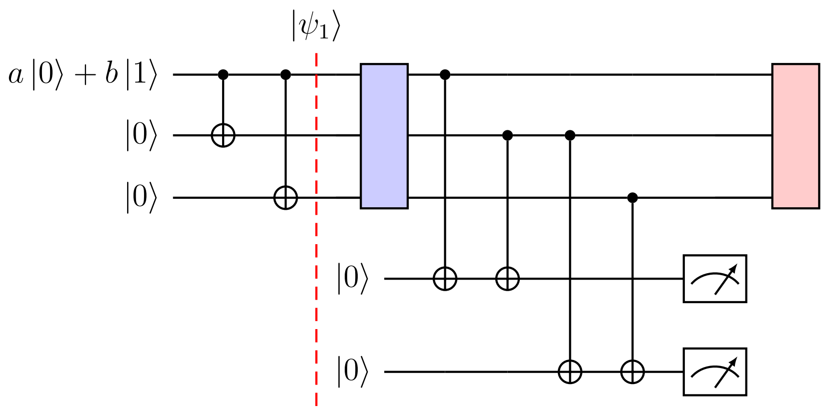
Tutorial 5



where the first measurement is a computational basis measurement of $Z_1 \otimes Z_2$, and the second measurement is a computational measurement of $Z_2 \otimes Z_3$. The blue box is hiding a (possible) error that has occurred, which will be nothing (identity) or a bit-flip error X that acts on any one of the three qubits. The pink box is hiding the corresponding correction.

a. What is the state $|\psi_1\rangle$?

Solution: The state is: $|\psi_1\rangle = a|000\rangle + b|111\rangle$

b. Assume that the outcomes of the measurements are: $s_1 = +1$ (for the upper detector measurement) and $s_2 = -1$ (for the lower detector measurement).

1. What does this indicate about the parity of the qubits just before the pink box?
2. Was an error occurring? If yes, where?
3. What was the associated correction operation?

Solution: Since $s_1 = +1$ and $s_2 = -1$, that means that we have the following projectors:
$P_{1,2}^{+1} = |00\rangle_{12}\langle 00|_{12} + |11\rangle_{12}\langle 11|_{12}$ and
$P_{2,3}^{-1} = |01\rangle_{23}\langle 01|_{23} + |10\rangle_{23}\langle 10|_{23}$. These will help us to answer the sub-questions:

i) Since we have $P_{1,2}^{+1}$ for the first and second qubit, this means that they have even parity and therefore that they have the same value. Similarly, since we have $P_{2,3}^{-1}$ for the second and third qubit, this means that they have odd parity, i.e. different values.

ii) In order to find P_i, the overall projector, we need to do the following calculation:

$$\begin{aligned}
&(P_{1,2}^{+1} \otimes I_3)(I_1 \otimes P_{2,3}^{-1}) \\
=&[(|00\rangle_{12}\langle 00|_{12} + |11\rangle_{12}\langle 11|_{12}) \otimes I_3][I_1 \otimes (|01\rangle_{23}\langle 01|_{23} + |10\rangle_{23}\langle 10|_{23})] \\
=&(|00\rangle_{12}\langle 00|_{12} + |11\rangle_{12}\langle 11|_{12}) I_1 \otimes I_3 (|01\rangle_{23}\langle 01|_{23} + |10\rangle_{23}\langle 10|_{23}) \\
=&|001\rangle_{123}\langle 001|_{123} + |110\rangle_{123}\langle 110|_{123} \\
P_3 =&|001\rangle\langle 001| + |110\rangle\langle 110|
\end{aligned}$$

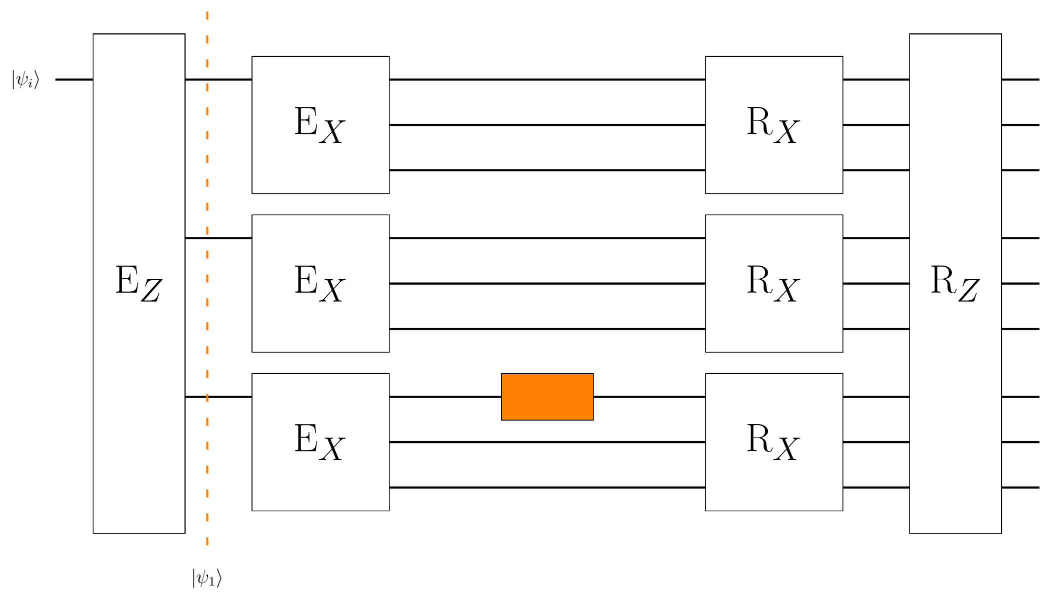
iii) Since the overall Projector is P_3, this means that there is a bit-flip error on the third qubit, and therefore the blue box is hiding an X gate on the third qubit, and likewise the recovery gate (under the pink box) will be an X gate on the third qubit, to correct the bit-flip error.

c. Why is it so important to check the parity of the qubits rather than measuring their outputs directly?

Solution: It is important to check the parity of the qubits to preserve the coherence of the system. We do not want the logical qubit encoded in the larger physical quantum state to collapse, as otherwise it would change from its initial state we want to preserve. We want to find enough partial information on the state of the large physical register in a clever way that it does not perturb the single logical qubit encoded into it.

Problem 3: Shor's 9-Qubit Code

Shor's 9 qubit code allows us to encode our state in 9 qubits and determine whether any arbitrary single qubit error has occured, and where. Consider the following circuit, as seen in the lectures:



where E_Z and E_X are the encoding circuits, and R_X and R_Z are the recovery circuits. The orange box in the circuit signifies a single qubit error occuring on the 7^{th} qubit. Assuming that the initial state is $|\psi_i\rangle = a|0\rangle + b|1\rangle$,

a. What is the state $|\psi\rangle_1$?

Solution: The state after the E_Z encoding is:
$|\psi_1\rangle = \alpha|+++\rangle + \beta|---\rangle$

b. What is the state after the error has occurred when the error in the orange box is:

1. an X error.
2. a Z error.
3. an XZ error.

Solution: Recall that a bit-flip X error takes $|0\rangle \to |1\rangle$, and $|1\rangle \to |0\rangle$ and a phase-flip Z error takes $|+\rangle \to |-\rangle$ and $|-\rangle \to |+\rangle$. $XZ = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$, which means that the XZ error takes $|0\rangle \to |1\rangle$ and $|1\rangle \to -|0\rangle$, incidentally in the lectures we were told that $Y = iXZ$ and we can clearly see that $XZ = \frac{1}{i}Y$. Let us also deduce the state after the bit-flip encodings E_X, wich we call $|\psi_2\rangle$:

$$|\psi_2\rangle = \frac{1}{\sqrt{2^3}}(\alpha(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)+ \\ \beta(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle))$$

We call our state after applying the single qubit error on the 7^{th} qubit $|\psi_3\rangle$ and we have:

i) X error:
$|\psi_3\rangle = \frac{1}{\sqrt{2^3}}(\alpha(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|100\rangle + |011\rangle) + \beta(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|100\rangle - |011\rangle))$

ii) Z error:
$|\psi_3\rangle = \frac{1}{\sqrt{2^3}}(\alpha(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle - |111\rangle) + \beta(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle + |111\rangle))$

iii) XZ error:
$|\psi_3\rangle = \frac{1}{\sqrt{2^3}}(\alpha(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|100\rangle - |011\rangle) + \beta(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|100\rangle + |011\rangle))$

c. In case 1,2 and 3 determine what the syndromes returned by the measurements will be.

Solution: As discussed in the lectures, there are eight syndromes with outcomes $\{s_1, \cdots, s_8\}$, $s_i \in \{-1, +1\}$ and we will recall them here:

With the above table, we must deduce whether the syndrome s_i is equal to $+1$ or -1 in each of the cases.

i) When we apply the X error on qubit 7 the outcomes will be: $\{+1, +1, +1, +1, -1, +1, +1, +1\}$ since there is only odd parity between qubits 7 and 8.

ii) When we apply the Z error on qubit 7 the outcomes will be: $\{+1, +1, +1, +1, +1, +1, +1, -1\}$ since we have a phase-flip error in the last triplet of qubits which changes the X-parity captured in the measurement of the second-half of the syndromes, s_8.

Syndrome	Label
$Z_1 \otimes Z_2$	s_1
$Z_2 \otimes Z_3$	s_2
$Z_4 \otimes Z_5$	s_3
$Z_5 \otimes Z_6$	s_4
$Z_7 \otimes Z_8$	s_5
$Z_8 \otimes Z_9$	s_6
$X_1 \otimes X_2 \otimes X_3 \otimes X_4 \otimes X_5 \otimes X_6$	s_7
$X_4 \otimes X_5 \otimes X_6 \otimes X_7 \otimes X_8 \otimes X_9$	s_8

iii) When we apply the XZ error on qubit 7 we have both the syndromes from the X-recovery part and the syndromes from the Z-recovery part in part i) and ii), and therefore the outcomes are: $\{+1, +1, +1, +1, -1, +1, +1, -1\}$.

d. For the XZ error occuring, what is the state after the layer of bit-flip decoding? How does this lead to the correct state during the phase-flip decoding, where the parities of the operators $X_1 \otimes ... \otimes X_6$ and $X_4 \otimes ... \otimes X_9$ are measured?

Solution: For the case where the XZ error occurs, we have the state:

$$|\psi_3\rangle = \frac{1}{\sqrt{2^3}}(\alpha(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|100\rangle - |011\rangle) + \beta(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|100\rangle + |011\rangle))$$

and after the first recovery process R_X, the bit-flip error will be detected, and an X gate on qubit 7 will be applied such that the bit-flip error is corrected, and we have:

$$|\psi_3\rangle' = \frac{1}{\sqrt{2^3}}(\alpha(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle - |111\rangle) + \beta(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle + |111\rangle))$$

Then we have the recovery process R_Z where the phase-flip error will be detected in the last triplet of qubits, and a Z gate on any of the qubits $7, 8, 9$ is applied such that the phase-flip error is corrected, and we have:

$$|\psi_3\rangle'' = \frac{1}{\sqrt{2^3}}(\alpha(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle) + \beta(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle))$$

which is exactly the state encoding we had at the beginning.

e. Suppose that instead an X error happens on qubit 1 and a Z error on qubit 4. Would our quantum error correcting code detect and correct both errors?

Solution: Yes. At the first recovery process the bit-flip error on the first qubit will be detected and will be corrected. Then, at the second recovery process, the phase flip error on qubit 4 will be detected and corrected. However, Shor's 9 qubit code cannot correct any *arbitrary* two qubit error.