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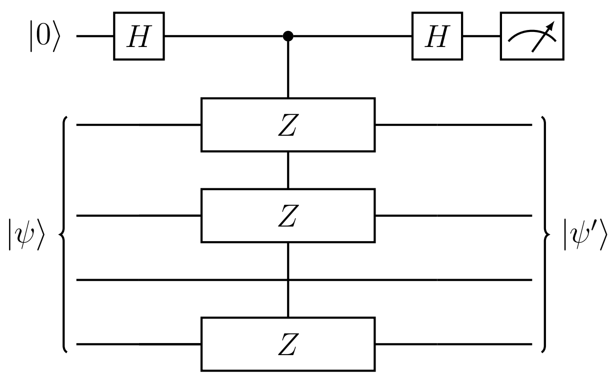
Tutorial 5

Problem 1: Three-Qubit Parity Check

We want to perform an even/odd parity check on qubits $1, 2, 4$. It's easy to see that the parity operator $P = Z \otimes Z \otimes I \otimes Z$ is both Hermitian and Unitary, so that it can both be regarded as an observable and a quantum gate. Suppose we wish to measure the observable P. That is, we desire to obtain a measurement result indicating one of the two eigenvalues, and leaving an updated state after the measurement that is projected to its corresponding eigenspace. We are going to show that the following circuit implements a measurement of P:

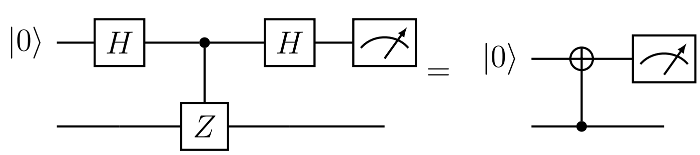


a. Derive the action of the three-qubit parity operator $P = Z \otimes Z \otimes I \otimes Z$ on the computational basis state $|x_1 x_2 x_3 x_4\rangle$. What are the eigenvalues of the operator P?

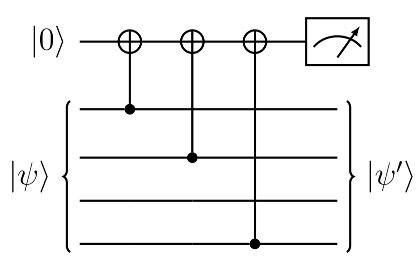
b. Derive the global state right before the measurement of the upper-qubit when the input state reads $|0\rangle \otimes |\psi\rangle$, where $|\psi\rangle = \sum_{x \in \{0,1\}^4} \gamma_x |x\rangle$ is a four qubit arbitrary input state and x is a four bit string.

c. Using the rules of partial measurement, show that the measurement of the upper-qubit projects the state of the lower four qubits to its odd or even parity subspaces, depending of the outcome being 0 or 1.

d. Prove that the two circuits below are equivalent:

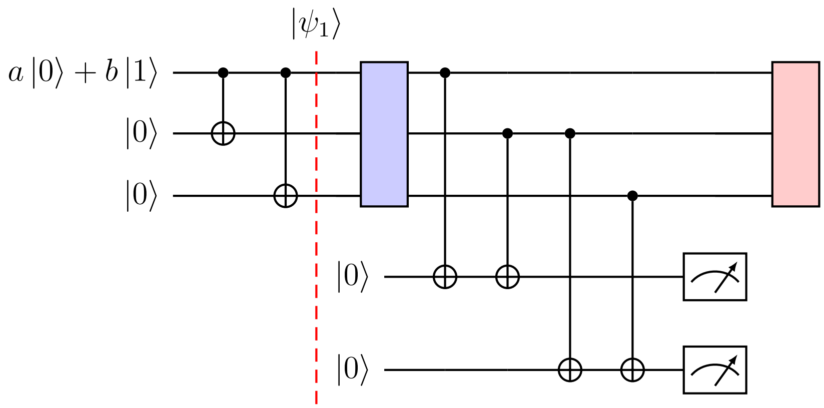


e. Prove that we can achieve the same result with the circuit:



Problem 2: Syndrome Measurement

Consider the following quantum error correcting circuit:



where the first measurement is a computational basis measurement of $Z_1 \otimes Z_2$, and the second measurement is a computational measurement of $Z_2 \otimes Z_3$. The blue box is hiding a (possible) error that has occurred, which will be nothing (identity) or a bit-flip error X that acts on any one of the three qubits. The pink box is hiding the corresponding correction.

a. What is the state $|\psi_1\rangle$?

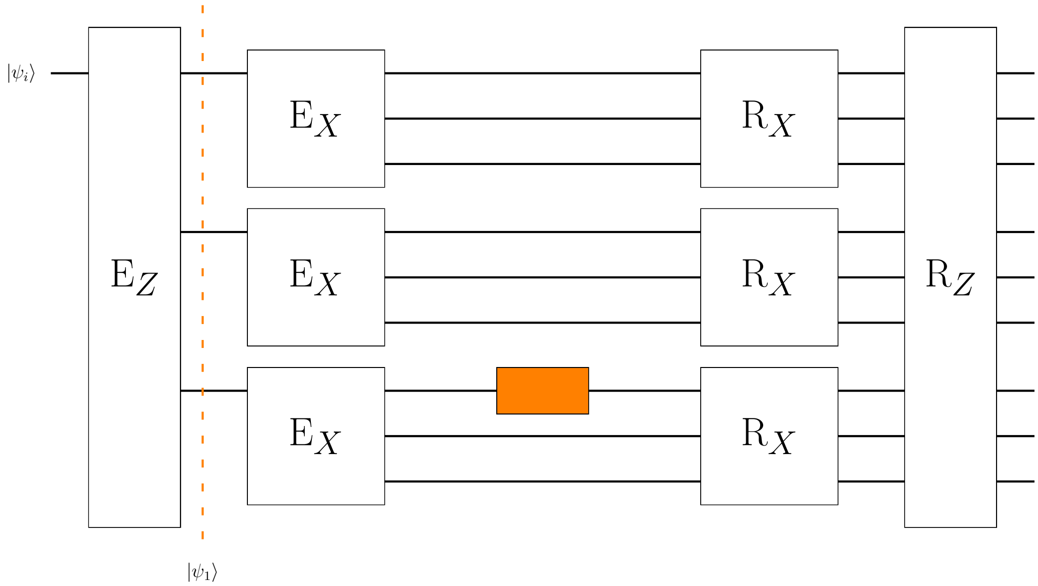
b. Assume that the outcomes of the measurements are: $s_1 = +1$ (for the upper detector measurement) and $s_2 = -1$ (for the lower detector measurement).

1. What does this indicate about the parity of the qubits just before the pink box?
2. Was an error occurring? If yes, where?
3. What was the associated correction operation?

c. Why is it so important to check the parity of the qubits rather than measuring their outputs directly?

Problem 3: Shor's 9-Qubit Code

Shor's 9 qubit code allows us to encode our state in 9 qubits and determine whether any arbitrary single qubit error has occured, and where. Consider the following circuit, as seen in the lectures:



where E_Z and E_X are the encoding circuits, and R_X and R_Z are the recovery circuits. The orange box in the circuit signifies a single qubit error occuring on the 7^{th} qubit. Assuming that the initial state is $|\psi_i\rangle = a|0\rangle + b|1\rangle$,

a. What is the state $|\psi\rangle_1$?

b. What is the state after the error has occurred when the error in the orange box is:

1. an X error.
2. a Z error.
3. an XZ error.

c. In case 1,2 and 3 determine what the syndromes returned by the measurements will be.

d. For the XZ error occuring, what is the state after the layer of bit-flip decoding? How does this lead to the correct state during the phase-flip decoding, where the parities of the operators $X_1 \otimes ... \otimes X_6$ and $X_4 \otimes ... \otimes X_9$ are measured?

e. Suppose that instead an X error happens on qubit 1 and a Z error on qubit 4. Would our quantum error correcting code detect and correct both errors?