Raul Garcia-Patron
Petros Wallden
Ioannis Kolotouros

Tutorial 6

IQC 2022-23
November 9, 2023

Problem 1: Quantum Fourier Transform

As you have seen in the lectures, we can represent any integer z in its binary form as:

$$z = z_1 z_2 \dots z_n$$

where $z_1, z_2, \dots, z_n$ are such so that:

$$z = z_n 2^{n-1} + \dots + z_2 2^1 + z_1$$

a. How many qubits at least would we need to encode the integer states $|14\rangle$ and $|9\rangle$? What is their binary representation when using qubits to encode the integers?

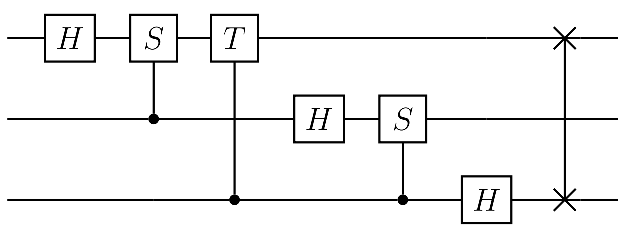
b. Recall that:

$$0.z_l z_{l+1} \dots z_m \equiv \frac{z_l}{2} + \frac{z_{l+1}}{2^2} + \dots + \frac{z_m}{2^{m-l+1}}$$

Calculate:

1. $2^3 0.z_1 z_2 z_3$, $2^2 0.z_1 z_2 z_3$ and $20.z_1 z_2 z_3$, where $z_i \in \{0, 1\}$.
2. $e^{2\pi i 2^2 0.j_1 j_2 j_3}$ where $j_i \in \{0, 1\}$.

c. Now consider the quantum Fourier circuit for three qubits:



with S and T being the gates:

$$S = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$

Suppose that we input the state $|j\rangle = |j_1 j_2 j_3\rangle$. What will be the output state?

Problem 2: Order-Finding

For two positive integers x and N with $x < N$ the *order* of x modulo N is defined to be the *least positive integer* such that:

$$x^r = 1 \mod N$$

Raul Garcia-Patron
Petros Wallden
Ioannis Kolotouros

Tutorial 6

a. Show that for $x = 2$ and $N = 5$ we have $r = 4$.

b. Now consider the transformation U which acts on the computational basis states as follows:

$$U_x |y\rangle \equiv |xy \mod N\rangle$$

Prove that:

1. $U_x U_{x'} = U_{xx'}$
2. $U_{x^{-1}} = U_x^{-1} = U_x^\dagger$, using the fat that x has an inverse $x^{-1} \pmod N$ if and only if x and N are co-prime.
3. $U_x U_x^\dagger = U_x^\dagger U_x = I$, which proves it is an unitary transformation.
4. $U_x^r = I$ where r is the period of x modulo N.

c. Show that the states:

$$|u_s\rangle \equiv \frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} e^{-\frac{2\pi isk}{r}} |x^k \mod N\rangle$$

for integer $0 \leq s \leq r-1$ are eigenstates of U_x. What is their corresponding eigenvalues?

d. As you can see preparing the state $|u_s\rangle$ requires that we know r in advance. Fortunately there is clever observation which circumvents the problems of preparing $|u_s\rangle$. Show that:

1. $$\sum_{s=0}^{r-1} e^{-2\pi isk/r} = r\delta_{k,0}$$
2. $$\frac{1}{\sqrt{r}} \sum_{s=0}^{r-1} e^{2\pi isk/r} |u_s\rangle = |x^k \mod N\rangle$$

which has as special case when $k = 0$:

$$\frac{1}{\sqrt{r}} \sum_{s=0}^{r-1} |u_s\rangle = |1\rangle ,$$

which is a trivial state to generate. This opens the door to applying quantum phase-estimation to sample from $\varphi = s/r$, which later leads to a guess of r as explained in the lecture on Shor's algorithm.

e. If we wanted to apply a phase estimation procedure we must have efficient procedures to implement a controlled-U^{2^j} operation for any integer j. Given an integer number x, propose a technique to compute x^{2^k} that scales linearly in k.

f. Assuming that we are given an unitary S such that implements $S|x\rangle = |x^2 \mod N\rangle$ that needs $O(L^2)$ gates, where $L = \lceil \log N \rceil$, i.e., the size of the register. How many gates we will be needed to implement $|x\rangle \rightarrow |x^{2^k} \mod N\rangle$?