Introduction to Quantum Computing
Lecture 24: MBQC II

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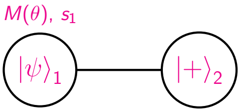
This Lecture

1. General Measurement Patterns and Background
2. How to cancel the "corrections" due to randomness
3. Output corrections, an Example and MBQC recap

General Measurement Patterns and Background

Previously in MBQC



The $J(-\theta)$-gate MBQC pattern

The above measurement pattern results to:

$$X^{s_1} J(-\theta) |\psi\rangle_2 = X^{s_1} HR(-\theta) |\psi\rangle_2$$

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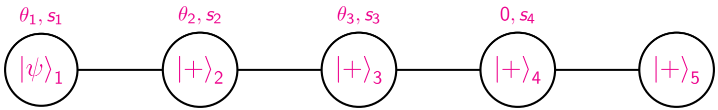

A general single-qubit gaete: MBQC pattern

The above measurement pattern results to:

$$X^{s_4} J(0) X^{s_3} J(-\theta_3) X^{s_2} J(-\theta_2) X^{s_1} J(-\theta_1) |\psi\rangle$$

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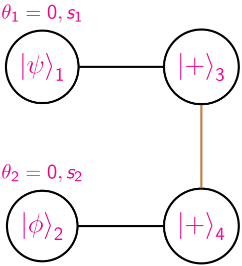

The $(\wedge Z_{12})\,(H_1 \otimes H_2)$-gate MBQC pattern

The above measurement pattern results to:

$$\wedge Z_{34}\,(X_3^{s_1} H_3 \,|\psi\rangle_3 \otimes X_4^{s_2} H_4 \,|\phi\rangle_4)$$

Measurement Pattern for Generic Computation

- Corrections appear when $s_i \neq 0$
- Let ϕ_i be the measurement angles that implement the desired unitary if **all** measurements give the result zero $s_i = 0 \ \forall \ i$
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- **Default angles determine computation:** $\{\phi_1, \phi_2, \cdots\}$
- **Corrected angles** are the one **used for measurements:** $\{\phi'_1, \phi'_2, \cdots\}$

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Definition: An entanglement graph (G, I, O), with I, O input resp output vertices, has flow if there exists a map $f : O^c \to I^c$ and a partial order $\preceq$ over qubits

1. $x \sim f(x)$: (x and $f(x)$ are neighbours in the graph)
2. $x \preceq f(x)$: ($f(x)$ is to the future of x with respect to the partial order)
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- **Example**: 2-dim lattice. $f(x) \Rightarrow$ same row, next column

Graph States as Stabiliser States

Graph state $|G\rangle$ is defined as:

$$|G\rangle = \prod_{(a,b)\in E} \wedge Z^{(a,b)} |+\rangle^{\otimes V}$$

An operator A **stabilises** a state if ($\lambda = +1$ eigenspace)

$$A|\psi\rangle = |\psi\rangle$$

The state $|G\rangle$ is a **stabiliser state** with generators:

$$K_i := X_i \left(\prod_{j\in N_G(i)} Z_j \right)$$

For each vertex $i \in V$ there is a stabiliser that has X at that vertex and Z to all its neighbours $N_G(i)$ in the graph.

Graph States as Stabiliser States

Operators K_i stabilise $|G\rangle$:

$$\begin{aligned} K_i|G\rangle &= X_i \left(\prod_{j\in N_G(i)} Z_j\right) \prod_{(a,b)\in E} \wedge Z^{(a,b)} |+\rangle^{\otimes V} \\ &= X_i \prod_{(a,b)\in E} \wedge Z^{(a,b)} |+\rangle^{\otimes V\setminus N_G(i)} |-\rangle^{\otimes j\in N_G(i)} \end{aligned}$$

where we used that Z commutes with $\wedge Z$ and that $Z|+\rangle = |-\rangle$.

We know that $X_i \wedge Z^{(i,j)} = \wedge Z^{(i,j)} X_i Z_j$ and we get:

$$K_i \left|G\right\rangle = \prod_{(a,b)\in E} \wedge Z^{(a,b)} \left(X_i \prod_{j\in N_G(i)} Z_j \right) \left|+\right\rangle^{\otimes V \setminus N_G(i)} \left|-\right\rangle^{\otimes j\in N_G(i)}$$

since X_i acts as above if i belongs to that edge while it commutes with all the other $\wedge Z$ that do not involve qubit i. However this changes back the states since $Z\left|-\right\rangle = \left|+\right\rangle$, and $X\left|+\right\rangle = \left|+\right\rangle$ results to

$$K_i \left|G\right\rangle = \prod_{(a,b)\in E} \wedge Z^{(a,b)} \left|+\right\rangle^{\otimes V}, \ \forall\ i \in V = \left|G\right\rangle$$

- It can be shown that this set of generators uniquely determines the graph state $\left|G\right\rangle$.

Part II

How to cancel the "corrections" due to randomness

How to apply an operator by acting on different qubits

Given a graph state $|G\rangle$, we can apply X, Z operators at qubit i by acting on qubits other than i.

We will use: $K_i |G\rangle = |G\rangle$

1. To apply X_i:

$$X_i |G\rangle = X_i K_i |G\rangle = \prod_{j \in N_G(i)} Z_j |G\rangle$$

   where $N_G(i)$ are the neighbours of i in the graph

2. To apply Z_i:

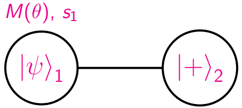
$$Z_i |G\rangle = Z_i K_{f(i)} |G\rangle = X_{f(i)} \prod_{j \in N_G(f(i)) \setminus i} Z_j |G\rangle$$

   where $f(i)$ is the flow of i and $N_G(f(i)) \setminus i$ are all the neighbours of $f(i)$ in the graph apart from i

How to apply an operator by acting on different qubits

Can cancel a correction **after** measuring that qubit, provided **all** qubits j involved are still not measured!

- This **cannot** be done for the X operator (qubits both to the past and future according to the definition of flow).
  Need to adapt the angles
- This **can** be done for the Z operator because of properties of the flow! (conditions 2 and 3)



Measuring at θ qubit 1

Recall (previous lecture) the state before measurement is:

$$\begin{aligned} |\chi\rangle_{12} &= \wedge Z_{12}\left(|\psi\rangle_1 \otimes |+\rangle_2\right) = a\,|0+\rangle_{12} + b\,|1-\rangle_{12} \\ &= |+_\theta\rangle_1\, X^{s_1=0} J(-\theta)_2\,|\psi\rangle_2 + |-_\theta\rangle_1\, X_2^{s_1=1} J(-\theta)_2\,|\psi\rangle_2 \end{aligned}$$

Rules for adapting measurement angles

If we could have started with $Z_1^{s_1} |\psi\rangle_1$ state instead of $|\psi\rangle_1$:

$$\begin{aligned} |\chi\rangle_{12} &= |+_\theta\rangle_1 X^{s_1} J(-\theta)_2 Z_2^{s_1} |\psi\rangle_2 + |-_\theta\rangle_1 X_2^{s_1} J(-\theta)_2 Z_2^{s_1} |\psi\rangle_2 \\ &= |+_\theta\rangle_1 J(-\theta)_2 |\psi\rangle_2 + |-_\theta\rangle_1 J(-\theta)_2 |\psi\rangle_2 \end{aligned}$$

using that $J(-\theta)Z^{s_1} = X^{s_1} J(-\theta)$ and $X^{s_1} X^{s_1} = \mathbf{I}$. Now, there is **no random correction** and any outcome of the measurement leads to the desired gate.

- Getting the "wrong" outcome $s_i = 1$ is as if a Z-correction on the initial state was applied, and could cancel it by applying another Z on that qubit.
- However, to do this we **need to know** s_1 which is the outcome of measuring qubit 1, and this (clearly) happens **after** the preparation of qubit 1.

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- If the "correction is on non-output qubits, instead of acting on them, we can modify the measurement angles:

$$M_i^{\phi_i} X = M_i^{-\phi_i} \ ; \ M_i^{\phi_i} Z = M_i^{\phi_i + \pi}$$

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This holds since:

$X \left|+_\phi\right\rangle = e^{i\phi} \left|+_{(-\phi)}\right\rangle$ and $Z \left|+_\phi\right\rangle = \left|-_\phi\right\rangle = \left|+_{(\phi+\pi)}\right\rangle$

E.g. $\left|+_\phi\right\rangle \left\langle +_\phi\right| X e^{i\phi} = \left|+_\phi\right\rangle \left\langle +_{-\phi}\right|$ and $\left|+_\phi\right\rangle \left\langle +_\phi\right| Z = \left|+_\phi\right\rangle \left\langle +_{\phi+\pi}\right|$

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       The set of vertices that induce X corrections to qubit i is denoted as $S_x(i) = \{f^{-1}(i)\}$.
    2. A Z-correction from all qubits $j \neq i$ that their flow $f(j)$ is neighbour to i.

       I.e. take each k of the neighbours of i and find $j := f^{-1}(k)$

       The set of vertices that induce Z corrections to qubit i is denoted as $S_z(i) = \{j \neq i \ : \ i \in N_G(f(j))\}$.

Rules for adapting measurement angles

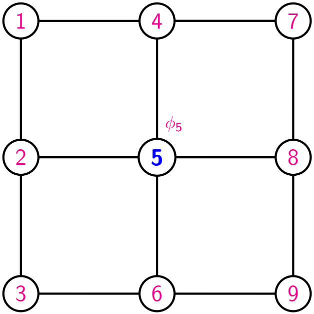
This sum leads to the corrected angle ϕ'_i:

$$\phi'_i = (-1)^{\sum_{j\in S_x(i)} s_j}\phi_i + \pi\left(\sum_{j\in S_z(i)} s_j\right)$$

$$\phi'_i = (-1)^{s_{f^{-1}(i)}}\phi_i + \pi\left(\sum_{j:i\in N_G(f(j))|j\neq i} s_j\right)$$

An Example, Output Corrections and MBQC recap

Example



Adapting measurement angles

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- Consider qubit 5:
  - $f^{-1}(5) = 2$
  - $N_G(5) = \{2, 4, 6, 8\}$
  - X-correction from s_2, i.e. $S_x(5) = \{2\}$
  - Z-corrections from: $\{s_1, s_3\}$, i.e. $S_z(5) = \{1, 3\}$. Since we look for $f^{-1}(\cdot)$ for each of the neighbours of 5. Qubit 2 has no past, qubit 8 has our qubit to its past, which leaves only the past of qubit 4 and qubit 6.

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  We then obtain the corrected measurement angle:

$$\phi'_5 = (-1)^{s_2}\phi_5 + \pi(s_1 + s_3)$$

Note: Depends on outcomes measured **before** qubit 5.

Corrections on Output Qubits

- **Note:** Some algorithms have no output qubits (end with measurement)
- For the rest algorithms, output qubits are treated differently
- Output qubits cannot be corrected with adapting the measurement angle
- There is an operation that needs to be applied. Conditional on previous outcomes, one applies a X operation, a Z operation, both or none.

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- There is an operation that needs to be applied. Conditional on previous outcomes, one applies a X operation, a Z operation, both or none.
- **Correction** for qubit i: $X_i^{s_{X,i}} Z_i^{s_{Z,i}}$

  $s_{X,i}$ is affected by qubits in $S_x(i)$ that is defined as for the measured qubits

  $s_{Z,i}$ is affected by qubits in $S_z(i)$ that is defined as for the measured qubits

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- The coefficient of the X correction of output qubit i, depends only on $f^{-1}(i)$
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- **NOTE.** Measured Vs Output qubits

  Same sets of $S_x(i)$ and $S_z(i)$ qubits that influence them

  Measured qubits modify their measurement angle to ϕ'_i

  Output qubits need to apply the extra gates Z^{s_Z} and X^{s_X}

MBQC Summary

- Start with a universal **graph state**
- The computation is performed by **measuring** one-by-one the qubits using **single-qubit** bases: $\{|+_\theta\rangle, |-_\theta\rangle\}$ (or $\{|0\rangle, |1\rangle\}$).
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- Exact measurement basis depends on previous outcomes.
- The **order of measurements** is determined by the **flow** $f(i)$ along with partial order $\preceq$.
- Single qubit gates performed using $J(\theta)$-gate and along with $\wedge Z$ is **universal**. Random corrections need to be cancelled.
- A unitary U is implemented by a set of **default** angles ϕ_i if all measurement had outcomes $s_i = 0$.
- The actual **"corrected" basis** that a qubit i is measured is modified, using flow & stabiliser properties:

$$\phi'_i = (-1)^{s_{f^{-1}(i)}}\phi_i + \pi\left(\sum_{j: i \in N_G(f(j))} s_j\right)$$

Further Reading on MBQC

1. One-way Quantum Computation - a tutorial introduction, D. Browne and H. Briegel, arxiv:quant-ph/0603226
2. An introduction to measurement based quantum computation, R. Jozsa, arxiv:quant-ph/0508124
3. Quantum computing with photons: introduction to the circuit model, the one-way quantum computer, and the fundamental principles of photonic experiments, S. Barz, Journal of Physics B: Atomic, Molecular and Optical Physics, Vol 48, Num. 8 (2015).
4. Chapter 7, Semantic Techniques in Quantum Computation – Editors Simon Gay and Ian Mackie